Local inference for the multifractional Brownian motion*

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May 14, 2025

Abstract

The fractional Brownian motion (fBm) is parameterized by the Hurst exponent $H \in (0,1)$, which determines the dependence structure and regularity of sample paths. Empirical findings suggest that for various phenomena, the Hurst exponent may be non-constant in time, giving rise to the so-called multifractional Brownian motion (mBm). The Itô-mBm is an alternative to the classical mBm, and has been shown to admit more intuitive sample path properties. In this paper, we show that Itô-mBm also allows for a simplified statistical treatment compared to the classical mBm. In particular, estimation of the local Hurst parameter $H(t)$ with Hölder exponent $\eta > 0$ achieves rates of convergence which are standard in nonparametric regression, whereas similar results for the classical mBm only hold for $\eta > 1$. Furthermore, we derive an estimator of the integrated Hurst exponent $\int_0^t H(s)\, ds$ which achieves a parametric rate of convergence, and use it to construct goodness-of-fit tests.

1 Introduction

Fractional Brownian motion (fBm) is a centered Gaussian process B_t^H with covariance function $\mathrm{Cov}(X_s, X_t) = \frac{1}{2}(|t|^{2h} + |s|^{2H} - |t-s|^{2H})$, for a scale parameter $\sigma^2 > 0$ and a so-called Hurst-exponent $H \in (0,1)$. It admits the Mandelbrot-van Ness representation

$$B_t^H = \int_{-\infty}^t \sigma\, [(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}]\, dB_s,$$

*The majority of this work was done while both authors were employed at RWTH Aachen University, Germany, funded by the Federal Ministry of Education and Research (BMBF) and the Ministry of Culture and Science of the German State of North Rhine-Westphalia (MKW) under the Excellence Strategy of the Federal Government and the Länder.

where B_s is a standard Brownian motion, and $(x)_+ = x \cdot \mathbb{1}_{x>0}$. The fBm and related processes have been widely applied in diverse fields. The Hurst exponent H is of major interest, as it governs the long-range dependence of the process, the regularity of sample paths, and the self-similarity.

In some applications, empirical evidence suggests that a single Hurst exponent H is not sufficient and a temporally-varying Hurst exponent H_t is more appropriate. A nonstationary generalizations of the fBm, the multifractional Brownian motion (mBm), has been introduced by Peltier and Lévy Véhel (1995) as

$$X_t = \int_{-\infty}^{t} \sigma \left[(t-s)_+^{H_t - \frac{1}{2}} - (-s)_+^{H_t - \frac{1}{2}}\right] dB_s. \tag{1}$$

See also Stoev and Taqqu (2006) for related definitions of mBm. Recently, Ayache et al. (2018) suggested an alternative nonstationary generalization of the fBm, given by

$$X_t = \int_{-\infty}^{t} \sigma_s \left[(t-s)_+^{H_s - \frac{1}{2}} - (-s)_+^{H_s - \frac{1}{2}}\right] dB_s. \tag{2}$$

Following Loboda et al. (2021), we refer to (1) as the classical mBm, and to (2) as Itô-mBm. Both processes behave locally like a fBm, i.e. $h^{-H_t}(X_{t+h} - X_t) \Rightarrow B_h^{H_t}$ as $h \to 0$. Thus, they are both valid candidate models for nonstationary extensions of the fBm. Probabilistic and statistical analysis of the classical mBm is facilitated by the fact that its covariance function admits an explicit expression (Stoev and Taqqu, 2006, Thm. 4.1). On the other hand, the Itô-mBm is well-defined as Itô integral if the Hurst exponent H_t is an (adapted) stochastic process. Moreover, the latter process has been shown in Loboda et al. (2021) to have attractive analytical features, see also Ayache and Bouly (2021). In particular, the regularity of its sample paths depends only on the values of H_t, whereas for the classical mBm, the regularity of $t \mapsto H_t$ is crucial as well. The goal of this paper is to demonstrate that the Itô-mBm is not only preferable from an analytical point of view, but also allows for improved statistical inference compared to the classical mBm.

Estimators for fBm from low-frequency observations $X_1, \ldots, X_n$ are reviewed and compared by Bardet (2018). Optimality of the MLE has been established by Cohen et al. (2013), and by Brouste and Fukasawa (2018) for the high-frequency regime $X_{1/n}, \ldots, X_1$. Local nonparametric estimators for the classical mBm (1) have be introduced by Coeurjolly (2005), Bertrand et al. (2013) and Bardet and Surgailis (2013). More recently, Shen and Hsing (2020) developed rate-optimal local nonparametric estimators of H_t exploiting higher-order smoothness. Lebovits and Podolskij (2017) estimate the global regularity $\min_t H_t$, and Bertrand et al. (2018) describe a goodness-of-fit test for the Hurst function. Further works on estimation of the mBm include, among others, Pianese et al. (2018); Garcin (2017); Bianchi et al. (2013). Estimation for some alternative non-stationary extensions of fBm allowing for irregular H_t is studied by Benassi et al. (2000) and Ayache and Lévy Véhel (2004); Ayache et al. (2007).

The limitation of all works on the classical mBm (1) is that they require $t \mapsto H_t$ to be Hölder continuous with exponent $\eta > \sup_t H_t$. This restriction is necessary to ensure a good local approximation of the process by a stationary fBm, see e.g. (Bardet and Surgailis, 2013, eqn. 7.14). It also emerges in the path properties of mBm, whose local Hölder coefficient at time t is $\min(\eta, H_t)$ (Herbin, 2006, Prop. 13). In contrast, prior work on the path properties of the Itô-mBm (2) has revealed that the lower bound on η is not necessary for this alternative multifractional extension of fBm, as the local Hölder exponent is H_t, regardless of the value of $\eta > 0$. The purpose of this article is to demonstrate that the Itô-mBm is not only attractive from an analytical perspective, but also statistically favorable. In particular, we show that a local estimator of H_t achieves standard nonparametric rates of convergence depending on η, regardless of $\max_t H_t$. In view of its elegant analytical and statistical properties, we propose the Itô-mBm as canonical nonstationary variant of the fBm.

To date, the only statistical treatment of the Itô-mBm is due to Ayache and Bouly (2023), showing uniform consistency of a localized Hurst estimator. Their estimator is analogous to the estimator we propose below, but the rate derived therein is slower, indicating that their asymptotic analysis is suboptimal. Here, we study two estimators $\widehat{H}_n(u)$ and $\widehat{H}_n^\dagger(u)$ based on local polynomial regression, which achieve the rate $n^{-\eta/(2\eta+1)}$ for $\eta \leq 1$ (exactly for $\widehat{H}_n^\dagger$, and up to logarithmic factors for $\widehat{H}_n$). This rate is standard in nonparametric estimation, and our results show that estimation of the Itô-mBm works *just as expected* – in stark contrast to estimation of the classical mBm, where estimators in the regime $\eta < 1$ admit non-standard rates of convergence (Bardet and Surgailis, 2013).

Beyond the local estimation of H_t, we derive an estimator $\widehat{\mathcal{H}}(u)$ of the integrated parameter $\mathcal{H}(u) = \int_0^u H_v \, dv$, and the error $\widehat{\mathcal{H}} - \mathcal{H}$ converges to a Gaussian process at rate $\sqrt{n}$. We use this estimator to construct a changepoint test for constant Hurst exponent, and goodness-of-fit test for the function $t \mapsto H_t$. An important feature of both tests is that they are robust to a non-constant volatility σ_t, which constitutes a nuisance under the null hypothesis. Our mathematical results are based on a functional central limit theorem for locally stationary time series established in Mies (2024), and the proof technique linking methods for stochastic processes in discrete and continuous time might be of independent interest.

We advocate for the use of Itô-mBm over the classical mBm on the basis of its attractive analytical and statistical properties. However, this is a purely mathematical argument and not based on empirical evidence in applications. Obviously, the looming open questions are: (i) Can we distinguish the two models based on data? (ii) Do the two models lead to different conclusions for practical questions, e.g. forecasting or asset pricing? We formulate these open questions as promising directions for future research, especially in view of the recent success of fractional models for stochastic volatility (Gatheral et al., 2018; Chong and Todorov, 2025).

The pointwise nonparametric estimator of the Hurst exponent is studied in Section 2. The integrated parameter estimator is introduced in Section 3, including the changepoint test and the goodness-of-fit test. All technical proofs are gathered in the Appendix.

2 Local nonparametric estimation

Performing high-frequency inference for the the Itô-mBm (2), we want to estimate the exponent H_s nonparametrically, using discrete observations $X_{\frac{1}{n}}, \ldots, X_{\frac{n}{n}}$. For simplicity of exposition, we suppose that $X_{-\frac{k}{n}}$ for $k = 1, 2, 3$ are also observable, and we define the second order increments as

$$\begin{aligned} \chi_{i,n} &= X_{\frac{i}{n}} - 2X_{\frac{i-1}{n}} + X_{\frac{i-2}{n}}, \qquad i = 1, \ldots, n, \\ \widetilde{\chi}_{i,n} &= X_{\frac{i}{n}} - 2X_{\frac{i-2}{n}} + X_{\frac{i-4}{n}}, \qquad i = 1, \ldots, n. \end{aligned}$$

Our estimation procedure is based on the change-of-frequency principle introduced by Kent and Wood (1997), which is a special case of the quadratic variation estimator of Istas and Lang (1997), and has also been used by Coeurjolly (2001): If $H_s = H$ and $\sigma_s = \sigma$ are constant, the self-similarity of the fBm yields $\mathbb{E}(\widetilde{\chi}_{i,n}^2)/\mathbb{E}(\chi_{i,n}^2) = 2^{2H}$, such that the moment estimator given by $\widehat{H} = \frac{1}{2}\log_2\left(\sum_i \widetilde{\chi}_{i,n}^2 / \sum_i \chi_{i,n}^2\right)$ is consistent at rate $\sqrt{n}$. For estimation of the Itô-mBm, we use the same idea, but localize the moment estimator around time $u \in (0,1)$. To this end, we use a kernel function $K : \mathbb{R} \to [0, \infty)$ satisfying

$$\begin{gathered} \int K(x)\,dx = 1, \qquad K(x) = 0 \text{ for } |x| > 1, \\ \{x : K(x) > 0\} \text{ has positive Lebesgue measure.} \end{gathered} \tag{K}$$

and a bandwidth $b = b_n > 0$, denote $K_b(x) = \frac{1}{b}K(\frac{x}{b})$. For a bandwidth $b = b_n > 0$, denote $K_b(x) = \frac{1}{b}K(\frac{x}{b})$, and for any $u \in (0,1)$, consider the locally weighted polynomial regression (Fan and Gijbels, 2018; Tsybakov, 2008)

$$\min_q \sum_{i=1}^n K_{b_n}(\tfrac{i}{n} - u) \left\| \begin{pmatrix} \chi_{i,n}^2 \\ \widetilde{\chi}_{i,n}^2 \end{pmatrix} - q(\tfrac{i}{n}) \right\|^2,$$

where the minimum q^* is determined among all polynomials $q : (0,1) \to \mathbb{R}^2$ of degree l We use $\widehat{\phi}_n(u) = q^*(u)$ as moment estimator at location u. It is well known (Tsybakov, 2008, Lem. 1.3 & Prop. 1.12) that the estimator admits the linear representation

$$\widehat{\phi}_n(u) = \sum_{i=1}^n w_{i,n}(u) \begin{pmatrix} \chi_{i,n}^2 \\ \widetilde{\chi}_{i,n}^2 \end{pmatrix},$$

and the weights satisfy, for some universal $C > 0$, and for all $u \in [b_n, 1 - b_n]$,

(i) $\sup_i |w_{i,n}(u)| \leq \frac{C}{nb_n}$,

(ii) $\sum_{i=1}^n |w_{i,n}(u)| \leq C$,

(iii) $w_{i,n}(u) = 0$ for $|\frac{i}{n} - u| > b_n$

(iv) $\sum_{i=1}^n w_{i,n}(u) = 1$ and $\sum_{i=1}^n (\frac{i}{n} - u)^k w_{i,n}(u) = 0$ for $k = 1, \ldots, l$.

Remark 1. If K is supported on $[-1,0]$ such that still $\{x \in [-1,0] : K(x) > 0\}$ has positive Lebesgue measure, then the latter properties (i)–(iv) hold for all $u \in [b_n, 1]$. Similarly, if K is supported on $[0,1]$, then (i)–(iv) hold for all $u \in [0, 1-b_n]$. Moreover, if $K(x) > K_0 \mathbb{1}(|x| < \tau)$ for some small $\tau > 0$ and $K_0 > 0$, then properties (i)–(iv) hold for all $u \in [0,1]$.

We estimate H_u via the log-ratio estimator

$$\widehat{H}_n(u) = \left(\frac{1}{2} \log_2 \left(\frac{\widehat{\phi}_{n,2}(u)}{\widehat{\phi}_{n,1}(u)} \right) \right) \vee 0 \ \wedge 1.$$

Theorem 1. *Suppose that $v \mapsto \theta_v = (\sigma_v, H_v)$ is Hölder continuous with exponent $\eta \in (0, l+1]$, and that $0 < \underline{H} \leq H_v \leq \overline{H} < 1$ for all v, and $0 < \underline{\sigma}^2 \leq \sigma_v^2 \leq \overline{\sigma}^2 < \infty$. If $b_n \ll n^{q-1}$ for some $q \in (0,1)$, then for any $p \geq 2$,*

$$\widehat{H}_n(u) = H_u + \mathcal{O}_{L_p}\left(\log(n)^{\lceil \eta \rceil} b_n^\eta + \frac{1}{\sqrt{n\, b_n}} \right) + \mathcal{O}\left(\log(n) n^{-(1 \wedge \eta)} \right).$$

The bound holds uniformly for $\theta \in [\underline{\sigma}^2, \overline{\sigma}^2] \times [\underline{H}, \overline{H}]$, and $u \in [b_n, 1-b_n]$.

The upper bound is minimized for $b_n \asymp n^{-\frac{1}{2\eta+1}} \log(n)^{-\frac{2\lceil \eta \rceil}{2\eta+1}}$, so that

$$\widehat{H}_n(u) = H_u + \mathcal{O}_{L_p}\left(\log(n)^{\frac{\lceil \eta \rceil}{2\eta+1}} n^{-\frac{\eta}{2\eta+1}} \right).$$

Remark 2. Under the conditions described in Remark 1, the bounds of Theorem 1 and Proposition 8 in the appendix can be extended to $u \in [b_n, 1]$, $u \in [0, 1-b_n]$, and $u \in [0,1]$, respectively.

For the locally constant variant, i.e. $l = 1$, this estimator is the same as considered by Coeurjolly (2005) for the classical mBm. Central to this approach is the scaling relation $\mathbb{E}(\widetilde{\chi}_{i,n}^2)/\mathbb{E}(\chi_{i,n}^2) \approx 2^{2H_{i/n}}$, and the estimator $\widehat{H}(u)$ utilizes this by estimating the mean first, and then taking the ratio. An alternative approach pursued by Shen and Hsing (2020) is to take the ratio first, and smooth second. This leads to the local polynomial estimator

$$\widehat{H}_n^\dagger(u) = \frac{1}{2} \sum_{i=1}^n w_{i,n}(u) \log_2 \left(\frac{\widetilde{\chi}_{i,n}^2}{\chi_{i,n}^2} \right).$$

Again, we may constrain the estimator manually to the interval $[0,1]$. It turns out that this estimator leads to an improvement of the rate of convergence by a logarithmic factor. Compared to the results of Shen and Hsing (2020) for the classical mBm, we show that this estimation approach, when applied to the Itô-mBm, also works for less smooth Hurst functions, $\eta < 1$.

Theorem 2. *Under the conditions of Theorem 1, the estimator $\widehat{H}_n^\dagger$ satisfies*

$$\widehat{H}_n^\dagger(u) = H_u + \mathcal{O}\left(b_n^\eta\right) + \mathcal{O}_{L_2}\left(\frac{1}{\sqrt{n\, b_n}} \right) + \mathcal{O}\left(\log(n) n^{-(1 \wedge \eta)} \right).$$

The upper bound is minimized for $b_n \asymp n^{-\frac{1}{2\eta+1}}$, so that

$$\widehat{H}_n^\dagger(u) = H_u + \mathcal{O}_{L_2}\left(n^{-\frac{\eta}{2\eta+1}}\right).$$

Remark 3. The estimator of Shen and Hsing (2020) achieves the rate $\mathcal{O}((n\log(n)^2)^{-\frac{\eta}{2\eta+1}})$. The logarithmic advantage can be attributed to the fact that they suppose σ to be constant in time. It may thus be estimated at a faster rate, and is essentially known for the local estimation of H_u, which allows for slightly better estimation of the latter. The same phenomenon of logarithmically faster rates if σ is known has been shown for the stationary case of the fBm (Brouste and Fukasawa, 2018).

Remark 4. The slower rate of the estimator $\widehat{H}_u$ is due to a logarithmically larger bias of the estimator $\widehat{\phi}_n(u)$. In particular, Lemma 3 in the appendix shows that $\mathbb{E}\chi_{i,n}^2 \approx n^{2H_{i/n}}\sigma_{i/n}^2\Gamma_{H_{i/n}}(0)$, and since H appears in the exponent, differentiating with respect to H to control the bias yields additional terms of order $\log(n)$.

3 Integrated parameter estimation

Another approach to treat the Hurst parameter nonparametrically is via the integrated parameter

$$\mathcal{H}(u) = \int_0^u H_v\, dv.$$

Interest in this functional arises because many hypotheses about H_u may be formulated in terms of $\mathcal{H}(u)$. For instance, the no-change hypothesis $H_v = H_0$ for all v is equivalent to $\mathcal{H}(u) = u\mathcal{H}(1)$; monotonicity of H_u is equivalent to convexity of $\mathcal{H}(u)$; and $H_v \geq \underline{H}$ is equivalent to $\mathcal{H}(u_2) - \mathcal{H}(u_1) \geq \underline{H}(u_2 - u_1)$ for all $u_2 \geq u_1$. We will discuss these examples in detail below.

A naive estimator for the integrated Hurst parameter $\mathcal{H}(u)$ can be obtained by integrating the local estimator, i.e. $\widetilde{\mathcal{H}}(u) = \int_0^u \widehat{H}_n(v)\, dv$. However, as discussed in (Mies, 2021), this estimator will in general be asymptotically bias-driven. Instead, we employ a generic linearization procedure which has been introduced in (Mies, 2021) in the context of locally-stationary time series. To this end, we propose the estimator

$$\widehat{\mathcal{H}}(u) = \frac{1}{n}\sum_{t=2L}^{\lfloor un \rfloor}\left\{\widehat{H}_n(\tfrac{t-L}{n}) + \frac{\widetilde{\chi}_{t,n}}{2\widehat{\phi}_{n,2}(\frac{t-L}{n})} - \frac{\chi_{t,n}}{2\widehat{\phi}_{n,1}(\frac{t-L}{n})}\right\},$$

where $\widehat{H}_n$ uses a kernel which is supported on $[-1,0]$. That is, $\widehat{H}_n(\frac{t}{n})$ only uses data up to time $\frac{t}{n}$.

Remark 5. If $t \leq \lceil nb_n \rceil$, the one-sided estimators $\hat{\phi}_n(\frac{t}{n})$ and $\widehat{H}_n(\frac{t}{n})$ effectively use the bandwidth $\frac{t}{n} \leq b_n$. This is due to the boundary adaptation of the local polynomial regression.

Theorem 3. *Suppose that $v \mapsto \theta_v = (\sigma_v, H_v)$ is Hölder continuous with exponent $\eta \in (\frac{1}{2}, 1]$, and that $0 < \underline{H} \le H_v \le \overline{H} < 1$ for all v, and $0 < \underline{\sigma}^2 \le \sigma_v^2 \le \overline{\sigma}^2 < \infty$. Suppose furthermore that b_n, L_n are chosen such that*

$$n^{-\frac{1}{2}+r} \ll b_n \ll n^{-\frac{1+2r}{4\eta}} \quad \text{for some } r \in (0, \tfrac{1}{2}),$$
$$n^{\frac{1}{6}-\frac{r}{3}} \ll L_n \ll n^{\frac{1}{2}-\delta} \quad \text{for some } \delta \in (0, \tfrac{1}{2}),$$
$$n^a \ll L_n \quad \text{for some } a > 0.$$

Then, as $n \to \infty$

$$\sqrt{n}(\widehat{\mathcal{H}}(u) - \mathcal{H}(u)) \quad \Rightarrow \quad \int_0^u \tau(H_v)\, dW_v \stackrel{d}{=} W\left(\int_0^u \tau^2(H_v)\, dv\right),$$

in the Skorokhod space $D[0,1]$. The local asymptotic variance $\tau^2(H)$ is defined in (9).

Remark 6. The condition on L_n is rather weak, and any choice $L_n \asymp n^a$ for some $a \in (\frac{1}{6}, \frac{1}{2})$ will satisfy the conditions. The choice of b_n basically needs to ensure that the local estimator $\widehat{\phi}_n(u)$ and hence $\widehat{H}_n(u)$, admits a sufficiently fast rate of convergence of order $\mathcal{O}(n^{-\frac{1}{4}-\frac{r}{2}})$. The feasible range of choices for b_n allows for this rate to be driven by bias or variance, and may always be satisfied for $\eta > \frac{1}{2}$.

We denote the limiting variance process by $\Sigma(u) = \int_0^u \tau^2(H_v)\, dv$. To perform statistical inference, we may estimate this process via the plug-in method as $\widehat{\Sigma}(u) = \frac{1}{n}\sum_{t=2L_n}^{\lfloor un \rfloor} \tau^2(\widehat{H}_n(\frac{t}{n}))$.

Theorem 4. *Under the conditions of Theorem 3,* $\sup_{u \in [0,1]} \left|\Sigma(u) - \widehat{\Sigma}(u)\right| \xrightarrow{\mathbb{P}} 0$.

That is, the limiting process $W(\Sigma(u))$ of Theorem 3 is approximated by $W(\widehat{\Sigma}(u))$, which allows for feasible statistical inference. In the next two subsections, we proceed to describe two specific hypothesis tests based on the estimator for the integrated Hurst parameter.

3.1 Testing for constant Hurst exponent

The estimator for the integrated Hurst exponent can be used to test for a constancy, that is, to treat the statistical problem

$$\mathbb{H}_0 : H_v \text{ constant} \quad \leftrightarrow \quad \mathbb{H}_1 : H_v \text{ not constant}.$$

Rejecting $\mathbb{H}_0$ is interpreted as evidence that a model based on fBm is not sufficient, and multifractional extensions need to be considered. This problem has been studied by Bibinger (2020). Therein, the specific multifractional model, i.e. Itô-mBm (2) vs classical mBm (1), does not matter, because the process is stationary under the null. Bibinger (2020) employs a CUSUM statistic based on the squared increments $\chi_{t,n}^2$. The test is applied to sunspot data, finding evidence for nonstationarity. A methodological limitation of the referenced method is that it will detect both changes in σ_v and in H_v, and a post-hoc analysis is necessary to distinguish both types of changes. In contrast, we would like to design a test which is only sensitive to changes in H_v, while being robust against changes in σ_v. That is, we treat the volatility as a nuisance. The relevance of allowing for nonstationary nuisance quantities under the no-change null hypothesis was first recognized by Zhou (2013), and further developed by Dette et al. (2019); Górecki et al. (2018); Pesta and Wendler (2020); Demetrescu and Wied (2018); Schmidt et al. (2021); Cui et al. (2021). To the best of our knowledge, changepoint testing with nonstationary nuisance quantities has not yet been considered for continuous-time processes.

To test for a change, we suggest the CUSUM-type statistic

$$T_{\text{CUSUM}}(\widehat{\mathcal{H}}) = \sup_{u\in[0,1]} \left|\widehat{\mathcal{H}}(u) - u\widehat{\mathcal{H}}(1)\right|.$$

Under $\mathbb{H}_0$, and if the volatility function satisfies the conditions of Theorem 3, the statistic $\sqrt{n}T(\widehat{\mathcal{H}})$ will converge in distribution to $\sup_{u\in[0,1]} |W(\Sigma(u)) - uW(\Sigma(1))|$. This can be used to derive critical values.

Proposition 1. *Suppose that the conditions of Theorem 3 hold. Let $q_n(\alpha)$ be the $1-\alpha$, X-conditional quantile of the random variable $Y_n = \sup_{u\in[0,1]} \left|W\left(\widehat{\Sigma}(u)\right) - uW(\widehat{\Sigma}(1))\right|$. If H_v is constant, then*

$$\limsup_{n\to\infty} P\left(\sqrt{n}\, T_{\text{CUSUM}}(\widehat{\mathcal{H}}) > q_n(\alpha)\right) \leq \alpha.$$

Alternatively, if H_v is not constant, then

$$\lim_{n\to\infty} P\left(\sqrt{n}\, T_{\text{CUSUM}}(\widehat{\mathcal{H}}) > q_n(\alpha)\right) = 1.$$

3.2 Application to goodness-of-fit testing

Let $\mathcal{G}_0 \subset \{H : [0,1] \to (0,1)\}$ be a class of candidate functions for the Hurst parameter, and we want to test the null hypothesis

$$\mathbb{H}_0 : H \in \mathcal{G}_0 \quad \leftrightarrow \quad \mathbb{H}_1 : H \notin \mathcal{G}_0.$$

For example, setting $\mathcal{G}_0 = \{H_v = av+b \,|\, b \in (0,1), a+b \in (0,1)\}$ yields a test for linearity, and setting $\mathcal{G}_0 = \{H \,|\, \exists v_0 \text{ s.t. } H \text{ is increasing on } [0,v] \text{ and decreasing on } [v,1]\}$ tests for unimodality. We suggest to apply the test statistic

$$\widehat{T}(\mathcal{G}_0) = \inf_{\tilde{H}\in\mathcal{G}_0} \widehat{T}(\tilde{H}), \quad \text{where } \widehat{T}(\tilde{H}) = \sup_{u\in[0,1]} \left|\widehat{\mathcal{H}}(u) - \int_0^u \tilde{H}(v)\, dv\right|.$$

Under $\mathbb{H}_0$, we clearly have $\sqrt{n}\widehat{T}(\mathcal{G}_0) \leq \sqrt{n}\widehat{T}(H) \Rightarrow \sup_{u\in[0,1]} \left|\int_0^u \tau(H_v)\, dW_v\right|$, and we can use the quantiles of the latter as critical values. Importantly, we can estimate the asymptotic variance function as in Theorem 4, which is also consistent under the alternative. As a result, we obtain a consistent test.

Proposition 2. *Suppose that the conditions of Theorem 3 hold. Let $q_n(\alpha)$ be the $1-\alpha$, X-conditional quantile of the random variable $Z_n = \sup_{u\in[0,1]} |W(\widehat{\Sigma}(u))|$. If $H \in \mathcal{G}_0$, then*

$$\limsup_{n\to\infty} P\left(\sqrt{n}\,\widehat{T}(\mathcal{G}_0) > q_n(\alpha)\right) \leq \alpha.$$

Alternatively, if $H \notin \mathcal{G}_0$ and $\mathcal{G}_0$ is closed with respect to the uniform norm, then

$$\lim_{n\to\infty} P\left(\sqrt{n}\,\widehat{T}(\mathcal{G}_0) > q_n(\alpha)\right) = 1.$$

An alternative goodness-of-fit test for the classical mBm and a singleton null $\mathcal{G}_0 = \{\tilde{H}\}$ has been suggested by Bertrand et al. (2018), based on a Cramer-von-Mises type test statistic of the form $\sum_{l=1}^{L} |\widehat{H}_n(\frac{l}{l}) - \tilde{H}(\frac{l}{L})|^2$ which they show to be asymptotically normal if $L = L_n \to \infty$ suitably as $n \to \infty$. However, they do not specify the constraints on L_n, making their test practically infeasible. Moreover, they suppose σ to be constant.

A Multiplier invariance principle

For the proof of Theorem 3, we make use of a functional central limit theorem for nonstationary time series, developed in Mies (2024). To make this article self-contained, we repeat the essential assumptions and the result in this section.

For iid random seeds $\epsilon_i \sim U(0,1)$, and functions $G_{t,n} : \mathbb{R}^\infty \to \mathbb{R}^d$, $t = 1, \ldots, n$, define the nonstationary array of time series $X_{t,n}$ as

$$X_{t,n} = G_{t,n}(\boldsymbol{\epsilon}_t), \quad t = 1, \ldots, n,$$
$$\boldsymbol{\epsilon}_t = (\epsilon_t, \epsilon_{t-1}, \ldots) \in \mathbb{R}^\infty.$$

Throughout this section, we assume that $\mathbb{E}(X_{t,n}) = 0$. For an independent copy $\tilde{\epsilon}_i \sim U(0,1)$, define also

$$\tilde{\boldsymbol{\epsilon}}_{t,h} = (\epsilon_t, \ldots, \epsilon_{t-h+1}, \tilde{\epsilon}_{t-h}, \epsilon_{t-h-1} \ldots) \in \mathbb{R}^\infty,$$
$$\boldsymbol{\epsilon}_{t,h} = (\epsilon_t, \ldots, \epsilon_{t-h+1}, \tilde{\epsilon}_{t-h}, \tilde{\epsilon}_{t-h-1} \ldots) \in \mathbb{R}^\infty.$$

We first impose the following set of assumptions, for some $\Gamma_n \geq 1$ and $\beta > 1$:

$$\|G_{1,n}(\boldsymbol{\epsilon}_0)\|_{L_2} + \sum_{t=2}^{n} \|G_{t,n}(\boldsymbol{\epsilon}_0) - G_{t-1,n}(\boldsymbol{\epsilon}_0)\|_{L_2} \leq \Theta_n \Gamma_n, \tag{A.1}$$

$$\max_{t=1,\ldots,n} \|G_{t,n}(\boldsymbol{\epsilon}_0) - G_{t,n}(\tilde{\boldsymbol{\epsilon}}_{0,h})\|_{L_q} \leq \Theta_n (h+1)^{-\beta}, \tag{A.2}$$

$$\int_0^1 \|G_{\lfloor vn \rfloor,n}(\boldsymbol{\epsilon}_0) - G_v(\boldsymbol{\epsilon}_0)\|_{L_2}\, dv \to 0. \tag{A.3}$$

For non-random matrices $g_{t,n}, g_u \in \mathbb{R}^{m\times d}$, and random matrices $\hat{g}_{t,n} \in \mathbb{R}^{m\times d}$, define

$$
\begin{aligned}
\Lambda_n &= \sqrt{\sum_{t=1}^{n} \|\hat{g}_{t,n} - g_{t,n}\|_{L_2}^2}, \\
\Psi_n &= \sum_{t=2}^{n} |g_{t,n} - g_{t-1,n}|, \\
\Phi_n &= \max_{t=1,\dots,n} |g_{t,n}| + \sup_{u \in [0,1]} |g_u|,
\end{aligned}
$$

and formulate the assumption

$$
\int_0^1 \|g_{\lfloor vn \rfloor, n} - g_v\| \, dv \to 0. \tag{A.4}
$$

Define the rate

$$
\chi(q, \beta) = \begin{cases} \frac{q-2}{6q-4}, & \beta \geq \frac{3}{2}, \\ \frac{(\beta-1)(q-2)}{q(4\beta-3)-2}, & 1 < \beta < \frac{3}{2}, \\ \frac{1}{2} - \frac{1}{\beta+1}, & 1 < \beta \leq \frac{2}{1+\frac{2}{q}}, \end{cases}
$$

and the local long run covariance matrices

$$
\begin{aligned}
\Sigma_{t,n} &= \sum_{h=-\infty}^{\infty} g_{t,n} \operatorname{Cov}[G_{t,n}(\boldsymbol{\epsilon}_0), G_{t,n}(\boldsymbol{\epsilon}_h)] g_{t,n}^T, \\
\Sigma_u &= \sum_{h=-\infty}^{\infty} g_u \operatorname{Cov}[G_u(\boldsymbol{\epsilon}_0), G_u(\boldsymbol{\epsilon}_h)] g_u^T.
\end{aligned}
$$

Theorem 5. *Suppose that $\hat{g}_{t,n}$ is $\boldsymbol{\epsilon}_{t-L}$-measurable for $L = L_n$, and let* (A.3), (A.1), (A.2), (A.4) *hold with $\Theta_n, \Phi_n = \mathcal{O}(1)$ such that*

$$
\begin{aligned}
n^{-\frac{1}{2}} \Lambda_n^2 + \Lambda_n L_n^{-\beta} + n^{\frac{1}{q} - \frac{1}{2}} L_n &\to 0 \\
(\Gamma_n + \Psi_n)^{\frac{\beta-1}{2\beta}} \sqrt{\log(n)} n^{-\xi(q,\beta)} &\to 0.
\end{aligned}
$$

Then

$$
\frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor un \rfloor} \hat{g}_{t,n} [X_{t,n} - \mathbb{E}(X_{t,n})] \Rightarrow \int_0^u \Sigma_v^{\frac{1}{2}} \, dW_v.
$$

B Proofs

B.1 Preliminaries

For $H \in (0, 1)$, define

$$
\gamma_H(h) = |h+1|^{2H} - 2|h|^{2H} + |h-1|^{2H}, h \in \mathbb{R},
$$

$$\Gamma_H(h) = -\gamma_H(h+1) + 2\gamma_H(h) - \gamma_H(h-1).$$

Note that $\gamma_H(h)$ is the autocovariance function of the increments of a fractional Brownian motion with Hurst parameter H, and $\Gamma_H(h)$ is the autocovariance of the corresponding second order increments. In particular, if $(\sigma_s, H_s) \equiv (\sigma, H)$, then $\mathrm{Cov}(\chi_{i,n}, \chi_{j,n}) = n^{-2H}\sigma^2\Gamma_H(i-j)$. Moreover, we note that $\Gamma_H(h) \asymp |h|^{2H-4}$ as $|h| \to \infty$.

We proceed to give quantitative bounds on the approximation error between the nonstationary process and its stationary tangent process. The following Lemma is in analogy to (Coeurjolly, 2005, Lemma 1) for the classical mBm. The notable difference is that we do not require $\eta > \sup_v H_v$, which is an advantage of the Itô-mBm model.

Lemma 3. *Suppose that $v \mapsto \theta_v = (\sigma_v, H_v)$ is Hölder continuous with exponent $\eta \in (0,1]$ on some open interval $I = (a,b] \subset (-\infty, 1]$, and that $0 < \underline{H} \le H_v \le \overline{H} < 1$ for all $v \in (-\infty, 1]$, and $\sigma_v^2 \le \overline{\sigma}^2 < \infty$. For any $\theta = (\sigma^2, H)$ and $\frac{i}{n} \in I$, it holds that*

$$n^{2H}\mathbb{E}\chi_{i,n}^2 = \sigma^2\Gamma_H(0) + \mathcal{O}\left(\log(n)n^{-\eta}\right) + \mathcal{O}\left(\|\theta - \theta_{\frac{i}{n}}\| \log(n) n^{2\|\theta-\theta_{\frac{i}{n}}\|}\right).$$

For any interval I, the bound holds uniformly for $\theta \in [0, \overline{\sigma}^2] \times [\underline{H}, \overline{H}]$.

Proof of Lemma 3. Denote $t_i = \frac{i}{n}$, and $\alpha_s = H_s - \frac{1}{2}$. Then, by Itô's formula,

$$\begin{aligned}
\mathbb{E}\chi_{i,n}^2 &= \int_{-\infty}^{t_i} \sigma_s^2 \left[(t_i - s)_+^{\alpha_s} - 2(t_{i-1}-s)_+^{\alpha_s} + (t_{i-2}-s)_+^{\alpha_s}\right]^2 ds \\
&= n^{-1}\int_{-\infty}^{n\cdot t_i} \sigma_{s/n}^2 \left[(t_i - \tfrac{s}{n})_+^{\alpha_{s/n}} - 2(t_{i-1} - \tfrac{s}{n})_+^{\alpha_{s/n}} + (t_{i-2} - \tfrac{s}{n})_+^{\alpha_{s/n}}\right]^2 ds \\
&= \int_{-\infty}^{i} n^{-2H_{s/n}}\sigma_{s/n}^2 \left[(i-s)_+^{\alpha_{s/n}} - 2(i-1-s)_+^{\alpha_{s/n}} + (i-2-s)_+^{\alpha_{s/n}}\right]^2 ds \\
&= \int_0^\infty g(v, \theta_{\frac{i}{n}-\frac{v}{n}})\, dv, \qquad \theta_v = (H_v, \sigma_v^2), \\
g(v,\theta) &= n^{-2H}\sigma^2 \left[v^{H-\frac{1}{2}} - 2(v-1)_+^{H-\frac{1}{2}} + (v-2)_+^{H-\frac{1}{2}}\right]^2.
\end{aligned}$$

Now observe that

$$\begin{aligned}
|\partial_\sigma g(v,\theta)| &\le n^{-2H}[v^{2H-1} \wedge v^{2H-5}], \\
|\partial_H g(v,\theta)| &\le n^{-2H}(16\sigma^2)[v^{2H-1} \wedge v^{2H-5}]\,[\log(n) + |\log(v)|].
\end{aligned}$$

We may thus bound

$$\begin{aligned}
&\int_0^\infty |g(v,\theta) - g(v, \theta_{\frac{i}{n}-\frac{v}{n}})|\, dv \\
&\le \int_0^\infty |g(v,\theta) - g(v,\theta_{\frac{i}{n}})|\, dv \; + \; \int_0^\infty |g(v,\theta_{\frac{i}{n}}) - g(v, \theta_{\frac{i}{n}-\frac{v}{n}})|\, dv \\
&= A_1 + A_2,
\end{aligned}$$

where, for some small $q \in (0,1)$,

$$A_1 \le C n^{-2(H \wedge H_{\frac{i}{n}})} \left[\log(n) |H - H_{\frac{i}{n}}| \;+\; |\sigma^2 - \sigma^2_{\frac{i}{n}}| \right],$$

$$A_2 \le \int_0^{n^{1-q}} |g(v, \theta_{\frac{i}{n}}) - g(v, \theta_{\frac{i}{n} - \frac{v}{n}})| \, dv \\ + \int_{n^{1-q}}^{\infty} |g(v, \theta_{\frac{i}{n}}) - g(v, \theta_{\frac{i}{n} - \frac{v}{n}})| \, dv.$$

Using the local Hölder continuity of $v \mapsto \theta_v$, we find that

$$\int_0^{n^{1-q}} |g(v, \theta_{\frac{i}{n}}) - g(v, \theta_{\frac{i}{n} - \frac{v}{n}})| \, dv \\ \le C \log(n) \max_{s \in [0, n^{-q}]} n^{-2H_{\frac{i}{n} - s}} \int_0^{n^{1-q}} \left(\tfrac{v}{n}\right)^{\eta} |v^{2\underline{H}-1} \wedge v^{2\overline{H}-5}|| \log(v)| \, dv \\ \le C \log(n) n^{-2H} n^{-\eta},$$

because $n^{n^{-\eta q}} \to 1$. Moreover,

$$\begin{aligned} \int_{n^{1-q}}^{\infty} |g(v, \theta_{\frac{i}{n}}) - g(v, \theta_{\frac{i}{n} - \frac{v}{n}})| \, dv &\le C \int_{n^{1-q}}^{\infty} \sup_{H \in [\underline{H}, \overline{H}]} n^{-2H} v^{2H-5} \, dv \\ &= \int_{n^{1-q}}^{n} \left(\tfrac{v}{n}\right)^{2\underline{H}} v^{-5} \, dv + \int_{n}^{\infty} \left(\tfrac{v}{n}\right)^{2\overline{H}} v^{-5} \, dv \\ &\le C(n^{(1-q)(2\underline{H}-4) - 2\underline{H}} + n^{-4}) \\ &\le C n^{-4+\epsilon}, \end{aligned}$$

for any $\epsilon > 0$, by making $q > 0$ sufficiently small. Since $\eta \in (0,1]$ and $H \in (0,1)$, the choice $\epsilon = \frac{1}{2}$ yields $n^{-4+\epsilon} \ll \log(n) n^{-2H-\eta}$.

Hence, we have shown that

$$\int_0^{\infty} g(v, \theta_{\frac{i}{n} - \frac{v}{n}}) \, dv = \int_0^{\infty} g(v, \theta) \, dv + \mathcal{O}\left(\log(n) n^{-2H-\eta}\right) + \mathcal{O}\left(\|\theta - \theta_{\frac{i}{n}}\| \log(n) n^{-2H + 2\|\theta - \theta_{\frac{i}{n}}\|}\right).$$

To complete the proof, we observe that $\int_0^{\infty} g(v, \theta) \, dv = n^{-2H} \sigma^2 \Gamma_H(0)$. □

Lemma 4. *Suppose that $v \mapsto \theta_v = (\sigma_v^2, H_v)$ is Hölder continuous with exponent $\eta \in (0,1]$ on some interval $I = (a,b] \subset (-\infty, 1]$, and that $0 < \underline{H} \le H_v \le \overline{H} < 1$ for all v, and $\sigma_v^2 \le \overline{\sigma}^2 < \infty$. Then for all $1 \le i \le j \le n$, such that $\frac{i}{n} \in I$,*

$$\operatorname{Cov}(\chi_{i,n}, \chi_{j,n}) = n^{-2H_{i/n}} \sigma^2_{i/n} \Gamma_{H_{i/n}}(i-j) \\ + \mathcal{O}\left(\log(n) n^{-2H_{i/n} - (\eta \wedge \frac{1}{2})} (|i-j| \vee 1)^{H_{i/n} - \frac{5}{2}}\right).$$

For any interval I, the bound holds uniformly for $\theta \in [0, \overline{\sigma}^2] \times [\underline{H}, \overline{H}]$.

Proof of Lemma 4. Denote $t_i = \frac{i}{n}$, and $\alpha_s = H_s - \frac{1}{2}$. By Itô's isometry,

$$\begin{aligned}
\mathrm{Cov}(\chi_{i,n}, \chi_{j,n}) &= \int_{-\infty}^{\frac{i}{n}} \sigma_s^2 \left[(t_i - s)^{\alpha_s} - 2(t_{i-1} - s)_+^{\alpha_s} + (t_{i-2} - s)^{\alpha_s}\right] \\
&\qquad \cdot \left[(t_j - s)^{\alpha_s} - 2(t_{j-1} - s)_+^{\alpha_s} + (t_{j-2} - s)^{\alpha_s}\right] ds \\
&= \int_{-\infty}^{i} n^{-1}\sigma_s^2 \left[(\tfrac{i}{n} - \tfrac{s}{n})^{\alpha_s} - 2(\tfrac{i-1}{n} - \tfrac{s}{n})_+^{\alpha_s} + (\tfrac{i-2}{n} - \tfrac{s}{n})^{\alpha_s}\right] \\
&\qquad \cdot \left[(\tfrac{j}{n} - \tfrac{s}{n})^{\alpha_s} - 2(\tfrac{j-1}{n} - \tfrac{s}{n})_+^{\alpha_s} + (\tfrac{j-2}{n} - \tfrac{s}{n})^{\alpha_s}\right] ds \\
&= \int_0^{\infty} n^{-2H_{\frac{i-v}{n}}} \sigma_s^2 \left[v^{\alpha_{\frac{i-v}{n}}} - 2(v-1)_+^{\alpha_{\frac{i-v}{n}}} + (v-2)^{\alpha_{\frac{i-v}{n}}}\right] \\
&\qquad \cdot \left[(v+\delta)^{\alpha_{\frac{i-v}{n}}} - 2(v+\delta-1)_+^{\alpha_{\frac{i-v}{n}}} + (v+\delta-2)^{\alpha_{\frac{i-v}{n}}}\right] dv \\
&=: \int_0^{\infty} f(v, \delta, \theta_{\frac{i-v}{n}})\, dv,
\end{aligned}$$

for $\delta = j - i$. We observe that, for $\delta \geq 1$,

$$\begin{aligned}
\|D_\theta f(v, \delta, \theta)\| &\leq C \log(n) n^{-2H} \left[v^{H-\frac{1}{2}} \wedge v^{H-\frac{5}{2}}\right] \cdot (v+\delta)^{H-\frac{5}{2}} (1 + |\log(v)|) \\
&\leq C \log(n) n^{-2H} (\delta \vee v)^{H-\frac{5}{2}} \left[v^{H-\frac{1}{2}} \wedge v^{H-\frac{5}{2}}\right] =: C \log(n) \overline{f}(v, \delta, H).
\end{aligned}$$

Using the η-Hölder continuity of $v \mapsto \theta_v$, with constant c, say, we find that

$$\begin{aligned}
&\left|\mathrm{Cov}(\chi_{i,n}, \chi_{j,n}) - \int_0^{\infty} f(v, \delta, \theta_{\frac{i}{n}})\, dv\right| \\
&\leq C \log(n) \int_0^{\infty} \left[\left(\tfrac{v}{n}\right)^{\eta} \wedge 1\right] \sup_{\substack{H \in [\underline{H}, \overline{H}] \\ |H - H_{i/n}| \leq c(v/n)^{\eta}}} \overline{f}(v, \delta, H)\, dv.
\end{aligned} \tag{3}$$

We split the domain of integration into the three segments $[0, 1]$, $[1, n^{1-q}]$, and $[n^{1-q}, \infty)$, for some small $q \in (0, 1)$ to be specified later.
The first portion of the integral may be bounded as

$$\begin{aligned}
&\int_0^1 \left[\left(\tfrac{v}{n}\right)^{\eta} \wedge 1\right] \sup_{\substack{H \in [\underline{H}, \overline{H}] \\ |H - H_{i/n}| \leq c(v/n)^{\eta}}} \overline{f}(v, \delta, H)\, dv \\
&\leq n^{-\eta - 2H_{i/n} + cn^{-\eta}} \int_0^1 v^{\underline{H} + \eta - \frac{1}{2}} \delta^{H_{i/n} + cn^{-\eta} - \frac{5}{2}} (1 + |\log(v)|)\, dv \\
&\leq C n^{-\eta - 2H_{i/n}}\, \delta^{H_{i/n} + cn^{-\eta} - \frac{5}{2}}.
\end{aligned}$$

The second portion of the integral may be bounded as

$$\int_1^{n^{1-q}} \left[\left(\tfrac{v}{n}\right)^{\eta} \wedge 1\right] \sup_{\substack{H \in [\underline{H}, \overline{H}] \\ |H - H_{i/n}| \leq c(v/n)^{\eta}}} \overline{f}(v, \delta, H)\, dv$$

$$\le n^{-\eta-2H_{i/n}+cn^{-q\eta}} \int_1^{n^{1-q}} v^{H_{i/n}+\eta-\frac{5}{2}+cn^{-q\eta}} \delta^{H_{i/n}-\frac{5}{2}+cn^{-q\eta}} (1+|\log(v)|)\, dv$$
$$\le n^{-\eta-2H_{i/n}} \delta^{H_{i/n}+cn^{-q\eta}-\frac{5}{2}} \left[1+n^{(1-q)(H_{i/n}+\eta-\frac{3}{2}+cn^{-q\eta})}\right]$$
$$\le n^{-\eta-2H_{i/n}} \delta^{H_{i/n}+cn^{-q\eta}-\frac{5}{2}} \left[1+n^{(1-q)(H_{i/n}+\eta-\frac{3}{2})}\right].$$

The third portion of the integral may be bounded as

$$\int_{n^{1-q}}^{\infty} \sup_{\substack{H\in[\underline{H},\overline{H}] \\ |H-H_{i/n}|\le c(v/n)^\eta}} \overline{f}(v,\delta,H)\, dv$$
$$\le \int_{n^{1-q}}^{\infty} \sup_{H\in[\underline{H},\overline{H}]} \overline{f}(v,\delta,H)\, dv$$
$$= \int_{n^{1-q}}^{\infty} \sup_{H\in[\underline{H},\overline{H}]} n^{-2H} v^{H-\frac{5}{2}} v^{H-\frac{5}{2}}\, dv$$
$$= \int_{n^{1-q}}^{\infty} \sup_{H\in[\underline{H},\overline{H}]} (\tfrac{v}{n})^{2H} v^{-5}\, dv$$
$$\le \int_{n^{1-q}}^{\infty} (\tfrac{v}{n})^{2\underline{H}} v^{-5}\, dv \;+\; \int_{n^{1-q}}^{\infty} (\tfrac{v}{n})^{2\overline{H}} v^{-5}\, dv$$
$$\le Cn^{(1-q)(2\underline{H}-4)-2\underline{H}} + Cn^{(1-q)(2\overline{H}-4)-2\overline{H}}$$
$$\le Cn^{-4(1-q)} \qquad \le C\delta^{H_{i/n}-\frac{5}{2}} n^{-2H_{i/n}-\epsilon}.$$

The last inequality holds for $\epsilon, q > 0$ sufficiently small. We have thus established that

$$\left|\mathrm{Cov}(\chi_{i,n},\chi_{j,n}) - \int_0^\infty f(v,\delta,\theta_{\frac{i}{n}})\, dv\right|$$
$$\le C\log(n) n^{-\eta-2H_{i/n}} \delta^{H_{i/n}-\frac{5}{2}+cn^{-q\eta}} \left[1+n^{(1-q)(H_{i/n}+\eta-\frac{3}{2})}\right] + Cn^{-4(1-q)},$$

and using that $\delta \in [1,n]$,

$$\le C\log(n)\delta^{H_{i/n}-\frac{5}{2}} n^{-2H_{i/n}} \left[n^{-\eta} + n^{(1-q)(\overline{H}-\frac{3}{2})}\right] + C(\tfrac{n}{\delta})^{\frac{5}{2}-H_{i/n}} n^{-4(1-q)}$$
$$\le C\log(n)\delta^{H_{i/n}-\frac{5}{2}} n^{-2H_{i/n}} \left[n^{-\eta} + n^{(1-q)(\overline{H}-\frac{3}{2})} + n^{\frac{5}{2}+H_{i/n}-4(1-q)}\right]$$
$$\le C\log(n)\delta^{H_{i/n}-\frac{5}{2}} n^{-2H_{i/n}} n^{-(\eta\wedge\frac{1}{2})}.$$

To conclude the proof, we observe that $\int_0^\infty f(v,\delta,\theta)\, dv$ is the lag-δ autocovariance of the second order increments at frequency $\frac{1}{n}$ of a fractional Brownian motion with parameters $\theta = (\sigma^2, H)$. Thus,

$$\int_0^\infty f(v,\delta,\theta)\, dv = n^{-2H}\sigma^2\, \Gamma_H(i-j).$$

In the derivations above, we assumed that $i > j$. For $i = j$, the claim is a direct consequence of Lemma 3, with $\theta = \theta_{i/n}$. □

Lemma 5. *Let the conditions of Lemma 4 hold, and define* $Z_{i,n} = (\chi_{i,n}^2,\ (\chi_{i,n}+2\chi_{i-1,n}+\chi_{i-2,n})^2)^T$. *Then*

$$\begin{aligned}\operatorname{Cov}(Z_{i,j}, Z_{j,n}) &= n^{-4H_{i/n}}\sigma_{i/n}^4 \Sigma_{H_{i_n}}(i-j) + \mathcal{O}\left(\log(n) n^{-4H_{i/n}-(\eta\wedge\frac{1}{2})}(|i-j|\vee 1)^{2H_{i/n}-5}\right)\\ &= \mathcal{O}\left(n^{-4H_{i/n}}(|i-j|\vee 1)^{-3}\right),\end{aligned}$$

where

$$\Sigma_H(h) := 2\begin{pmatrix}\Gamma_H(h)^2 & (\Gamma_H(h)+\Gamma_H(h+1))^2\\ (\Gamma_H(h)+\Gamma_H(h-1))^2 & (2\Gamma_H(h)+\Gamma_H(h-1)+\Gamma_H(h+1))^2\end{pmatrix}, \quad h \in \mathbb{Z}.$$

The bound holds uniformly for $\theta \in [0, \overline{\sigma}^2] \times [\underline{H}, \overline{H}]$.

Proof of Lemma 5. Define $Y_{i,n} = (\chi_{i,n},\ \chi_{i,n}+2\chi_{i-1,n}+\chi_{i-2,n})^T$, and introduce the matrix.

$$\overline{\Gamma}_H(h) := \begin{pmatrix}\Gamma_H(h) & \Gamma_H(h)+\Gamma_H(h+1)\\ \Gamma_H(h)+\Gamma_H(h-1) & 2\Gamma_H(h)+\Gamma_H(h-1)+\Gamma_H(h+1)\end{pmatrix}.$$

Via Lemma 4, we find that

$$\operatorname{Cov}(Y_{i,n}, Y_{j,n}) = n^{-2H_{i/n}}\sigma_{i/n}^2 \overline{\Gamma}_{H_{i/n}}(i-j) + \mathcal{O}\left(\log(n) n^{-2H_{i/n}-(\eta\wedge\frac{1}{2})}(|i-j|\vee 1)^{H_{i/n}-\frac{5}{2}}\right).$$

Note that the matrix $\Sigma_H(h)$ is twice the entry-wise square of $\overline{\Gamma}_H(h)$, and that $|\overline{\Gamma}_H(h)| \asymp h^{2H-4}$. Hence, Lemma 6 yields

$$\operatorname{Cov}(Z_{i,j}, Z_{j,n}) = n^{-4H_{i/n}}\sigma_{i/n}^4 \Sigma_{H_{i_n}}(i-j) + \mathcal{O}\left(\log(n) n^{-4H_{i/n}-(\eta\wedge\frac{1}{2})}(|i-j|\vee 1)^{2H_{i/n}-5}\right).$$

□

Lemma 6. *For two centered, jointly Gaussian random variables* X, Y, *it holds* $\operatorname{Cov}(X^2, Y^2) = \operatorname{Cov}(X,Y)^2 \frac{\operatorname{Var}(X^2)}{\operatorname{Var}(X)^2} = 2\operatorname{Cov}(X,Y)^2$.

Proof. Write $(X, Y) = (X, aX + Z)$ for $a = \operatorname{Cov}(X, Y)$, and Z centered Gaussian, independent form X. Observe that for independent centered Gaussian random variables X, Z, and $a \in \mathbb{R}$, we have $\operatorname{Cov}(X^2, (aX+Z)^2) = a^2\operatorname{Var}(X^2) + 2a\mathbb{E}(X^3Z) + \operatorname{Cov}(X^2, Z^2) = a^2\operatorname{Var}(X^2)$, and $\operatorname{Cov}(X, aX+Z) = a\operatorname{Var}(X)$. □

Lemma 7. *There exists a universal* K *such that for any two centered, jointly Gaussian random variables* X, Y,

$$\left|\operatorname{Cov}(\log(X^2), \log(Y^2))\right| \le K|\rho|.$$

Proof. Since this specific covariance is invariant to rescaling of X and Y, we may assume both are standard normal, with correlation ρ. We proceed similar to the proof of (Shen and Hsing, 2020, Lemma 8.5).

Denote by $H_l : \mathbb{R} \to \mathbb{R}$ the l-th Hermite polynomial (i.e. of degree l), and decompose $\log(x^2) = \sum_{l=0}^{\infty} c_l H_l(x)$. Because $\log(X^2) \in L_2(P)$, the sequence c_l is square-summable. Moreover,

$$\operatorname{Cov}(\log(X^2), \log(Y^2)) = \sum_{k,l=0}^{\infty} c_l c_k \operatorname{Cov}(H_l(X), H_k(Y)).$$

Now use $\operatorname{Cov}(H_l(X), H_k(Y)) = \rho^k k! \mathbb{1}(l = k)$ (Shen and Hsing, 2020, S.3.4) to find that

$$\operatorname{Cov}(\log(X^2), \log(Y^2)) = \sum_{k=1}^{\infty} c_k^2 \rho^k k!.$$

Because $\operatorname{Cov}(\log(X^2), \log(X^2)) < \infty$, corresponding to $\rho = 1$, we conclude that $\sum_k c_k^2 k! < \infty$. This yields $|\operatorname{Cov}(\log(X^2), \log(Y^2))| \leq |\rho| \sum_k c_k^2 k!$.

□

B.2 Local nonparametric estimation

The error of $\widehat{\phi}_n(u)$ admits the following asymptotic representation.

Proposition 8. *Suppose that $v \mapsto \theta_v = (\sigma_v, H_v)$ is Hölder continuous with exponent $\eta \in (0, l+1]$, and that $0 < \underline{H} \leq H_v \leq \overline{H} < 1$ for all v, and $\sigma_v^2 \leq \overline{\sigma}^2 < \infty$. If $b_n \ll n^{q-1}$ for some $q \in (0,1)$, then for any $p \geq 2$,*

$$n^{2H_u} \widehat{\phi}_n(u) = \sigma_u^2 \Gamma_{H_u}(0) \cdot \begin{pmatrix} 1 \\ 2^{2H_u} \end{pmatrix} + \mathcal{O}\left(\log(n)^{\lceil \eta \rceil} b_n^{\eta}\right) + \mathcal{O}_{L_p}\left(\frac{1}{\sqrt{n\, b_n}}\right) + \mathcal{O}\left(\log(n) n^{-(1\wedge\eta)}\right)$$

For any p, the bound holds uniformly for $\theta \in [0, \overline{\sigma}^2] \times [\underline{H}, \overline{H}]$, and for all (b_n, u) such that $u \in [b_n, 1 - b_n]$.

Proof of Proposition 8. Lemma 3 yields

$$\begin{aligned} \mathbb{E}\left(\widehat{\phi}_{n,1}(u)\right) &= \sum_{i=1}^{n} w_{i,n} \mathbb{E}\chi_{i,n}^2 = \sum_{i=1}^{n} w_{i,n} n^{-2H_{i/n}} \sigma_{i/n}^2 \Gamma_{H_{i/n}}(0) + \mathcal{O}\left(\log(n) n^{-(1\wedge\eta)-2H_u}\right), \\ \rightsquigarrow \quad \mathbb{E}\left(n^{2H_u} \widehat{\phi}_{n,1}(u)\right) &= \sum_{i=1}^{n} w_{i,n} n^{2(H_u - H_{i/n})} \sigma_{i/n}^2 \Gamma_{H_{i/n}}(0) + \mathcal{O}\left(\log(n) n^{-(1\wedge\eta)}\right). \end{aligned}$$

A Taylor expansion of the function $\mu_n(s) = n^{2(H_u - H_s)} \sigma_s^2 \Gamma_{H_s}(0)$, together with the properties of the weights, yields,

$$\mathbb{E}\left(n^{2H_u} \widehat{\phi}_{n,1}(u)\right) = \mu_n(u) + \mathcal{O}\left(b_n^{\eta} \log(n)^{\lceil \eta \rceil}\right) + \mathcal{O}\left(\log(n) n^{-(1\wedge\eta)}\right)$$

The same bound holds for $\widehat{\phi}_{n,2}$, hence

$$\mathbb{E}\left(n^{2H_u} \widehat{\phi}_n(u)\right) = \sigma_u^2 \Gamma_{H_u}(0) \cdot \begin{pmatrix} 1 \\ 2^{2H_u} \end{pmatrix} + \mathcal{O}\left(\log(n) n^{-(1\wedge\eta)}\right) + \mathcal{O}\left(b_n^{\eta} \log(n)^{\lceil \eta \rceil}\right),$$

and the bound holds uniformly in $u \in (0,1)$.

To bound the stochastic term, we use Lemma 5 which yields

$$\begin{aligned}
\left\| \operatorname{Cov}\left(n^{2H_u} \widehat{\phi}_n(u)\right) \right\| &\leq n^{4H_u} \sum_{i,j=1}^{n} w_{i,n}(u) w_{j,n}(u) \left\| \operatorname{Cov}(\delta_{i,n}, \delta_{j,n}) \right\| \\
&\leq \frac{C^2}{n^2\, b_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{1}_{|\frac{i}{n}-u|\leq b_n} \mathbb{1}_{|\frac{j}{n}-u|\leq b_n} \| \operatorname{Cov}(Z_{i,n}, Z_{j,n}) \|.
\end{aligned}$$

For $|\frac{i}{n} - u| \leq b_n \ll n^{1-q}$, and $|\frac{j}{n} - u| \ll n^{1-q}$, by virtue of Lemma 5,

$$\| \operatorname{Cov}(Z_{i,n}, Z_{j,n}) \| \leq C(|i-j| \vee 1)^{-3} n^{-4H_u}.$$

Thus,

$$\begin{aligned}
\left\| \operatorname{Cov}\left(n^{2H_u} \widehat{\phi}_n(u)\right) \right\| &\leq C \frac{1}{n^2 b_n^2} \sum_{i,j=1}^{n} \mathbb{1}_{|\frac{i}{n}-u|\leq b_n} \mathbb{1}_{|\frac{j}{n}-u|\leq b_n} (|i-j| \vee 1)^{-3} \\
&\leq C \frac{1}{n^2 b_n^2} \sum_{i=\lfloor n(u-b_n)\rfloor \vee 1}^{\lceil n(u+b_n)\rceil \wedge n} \sum_{j=-\infty}^{\infty} (|i-j| \vee 1)^{-3} \qquad \leq \frac{C}{n\, b_n}.
\end{aligned}$$

This establishes the upper bound on the stochastic term of $\widehat{\phi}_n(u)$ in L_2. To obtain the bound in L_p, we observe that $\widehat{\phi}_n(u)$ belongs to the second Wiener chaos, such that all L_p norms are equivalent (Nourdin and Peccati, 2012, 2.8.14).

□

Proof of Theorem 1. Proposition 8 yields

$$\begin{aligned}
\widehat{H}_n(u) &= \frac{1}{2} \log_2 \left(\frac{2^{2H_u} \sigma^2 \Gamma_{H_u}(0) + \mathcal{O}(\log(n)^{\lceil \eta \rceil} b_n^{\eta}) + \mathcal{O}_{L_p}(1/\sqrt{n b_n})}{\sigma^2 \Gamma_{H_u}(0) + \mathcal{O}(\log(n)^{\lceil \eta \rceil} b_n^{\eta}) + \mathcal{O}_{L_p}(1/\sqrt{n b_n})} \right) \\
&= \frac{1}{2} \log_2 \left(2^{2H_u} + \mathcal{O}(\log(n)^{\lceil \eta \rceil} b_n^{\eta}) + \mathcal{O}_{L_p}(1/\sqrt{n b_n}) \right) \\
&= H_u + \mathcal{O}_{L_p} \left(\log(n)^{\frac{\lceil \eta \rceil}{2\eta+1}} n^{-\frac{\eta}{2\eta+1}} \right).
\end{aligned}$$

□

Proof of Theorem 2. Denote $a_{i,n} = \mathbb{E}\chi^2_{i,n}$ and $\tilde{a}_{i,n} = \mathbb{E}\widetilde{\chi}^2_{i,n}$. Then $\chi^2_{i,n} \overset{d}{=} a_{i,n} Z^2$ for $Z \sim \mathcal{N}(0,1)$, and $\widetilde{\chi}^2_{i,n} \overset{d}{=} \tilde{a}_{i,n} Z^2$, so that Lemma 3 yields

$$\begin{aligned}
\mathbb{E} \log_2 \left(\tfrac{\widetilde{\chi}^2_{i,n}}{\chi^2_{i,n}} \right) &= \log_2 \left(\tfrac{\tilde{a}_{i,n}}{a_{i,n}} \right) + \left[\mathbb{E} \log_2(Z^2) - \mathbb{E} \log_2(Z^2) \right] = \log_2 \left(\tfrac{\tilde{a}_{i,n}}{a_{i,n}} \right) \\
&= 2H_{i/n} + \log_2 \left(\tfrac{(n/2)^{2H_{i/n}} \tilde{a}_{i,n}}{n^{2H_{i/n}} a_{i,n}} \right)
\end{aligned}$$

$$
\begin{aligned}
&= 2H_{i/n} + \log_2\left(\frac{\sigma^2\Gamma_{H_{i/n}}(0) + \mathcal{O}(\log(n)n^{-(\eta\wedge 1)})}{\sigma^2\Gamma_{H_{i/n}}(0) + \mathcal{O}(\log(n)n^{-(\eta\wedge 1)})}\right)\\
&= 2H_{i/n} + \log_2\left(1 + \mathcal{O}(\log(n)n^{-(\eta\wedge 1)})\right)\\
&= 2H_{i/n} + \mathcal{O}\left(\log(n)n^{-(\eta\wedge 1)}\right).
\end{aligned}
$$

Thus, by standard bias bounds for local polynomial estimators, we obtain the bias bound $\mathbb{E}(\widehat{H}_n^{\dagger}(u)) = H(u) + \mathcal{O}(b_n^{\eta} + \log(n)n^{-(\eta\wedge 1)})$.

To control the variance, we use Lemma 7 to find that

$$
\mathrm{Var}\left(\sum_{i=1}^{n} w_{i,n}(u)\log_2(\chi_{i,n}^2)\right) \leq K\sum_{i,j=1}^{n} |w_{i,n}(u)w_{j,n}(u)| \cdot |\mathrm{Cor}(\chi_{i,n}, \chi_{j,n})|.
$$

In the following, K is a generic constant which may vary from line to line. Lemma 4 yields

$$
|\mathrm{Cor}(\chi_{i,n}, \chi_{j,n})| \leq K|i-j|^{2\overline{H}-4} + K|i-j|^{\overline{H}-\frac{5}{2}} \quad \leq K|i-j|^{-\frac{3}{2}},
$$

which is summable. In combination with the boundedness and finite support of the weights $w_{i,n}(u)$, we find

$$
\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{n} w_{i,n}(u)\log_2(\chi_{i,n}^2)\right) &\leq K\sum_{i,j=\lfloor n\rfloor(u-b_n)\rfloor}^{\lceil n(u+b_n)\rceil} w_{i,n}(u)w_{j,n}(u)(|i-j|+1)^{-\frac{3}{2}}\\
&\leq \frac{K}{nb_n}\sum_{h=-\infty}^{\infty}(|h|+1)^{-\frac{3}{2}} \quad = \mathcal{O}(1/(nb_n)).
\end{aligned}
$$

Similarly, $\mathrm{Var}\left(\sum_{i=1}^{n} w_{i,n}(u)\log_2(\chi_{i,n}^2)\right) = \mathcal{O}(1/(nb_n))$, which in particularly yields

$$
\mathrm{Var}\left(\widehat{H}_n^{\dagger}(u)\right) = \mathcal{O}\left(\tfrac{1}{nb_n}\right).
$$

□

B.3 Integrated parameter estimation

Proof of Theorem 3. Step (i): Denote $t_0 = 2L_n$. We may write the estimator $\widehat{\mathcal{H}}(u)$ equivalently as

$$
\widehat{\mathcal{H}}(u) = \frac{1}{n}\sum_{t=t_0}^{\lfloor un\rfloor}\left\{m(\hat{\psi}_{t,n}) + Dm(\hat{\psi}_{t,n})\cdot(Z_{t,n} - \hat{\psi}_{t,n})\right\},
$$

with

$$
m : (0,\infty)^2 \to \mathbb{R},\ (x,y) \mapsto \left[\tfrac{1}{2}\log(\tfrac{y}{x})\right] \ \vee 0\ \wedge 1,
$$

$$Z_{t,n} = \frac{n^{2H_{t/n}}}{\sigma_{t/n}^2}\delta_{t,n}, \qquad \psi_u = \Gamma_{H_u}(0) \cdot \begin{pmatrix} 1 \\ 2^{2H_u} \end{pmatrix}, \qquad \psi_{t,n} = \psi_{t/n}$$

$$\hat{\psi}_{t,n} = \frac{n^{2H_{t/n}}}{\sigma_{t/n}^2}\widehat{\phi}(\tfrac{t-L_n}{n}) \text{ for } t = t_0, \ldots, n, \text{ and } \hat{\psi}_{t,n} = \psi_{t,n} \text{ for } t = 1, \ldots, t_0 - 1.$$

Moreover, using the L_q bound of Lemma 8, we have for any $q \geq 2$ and $t = L + \lceil nb_n \rceil, \ldots, n$,

$$\|\hat{\psi}_{t,n} - \psi_{t-L,n}\|_{L_q}^2 = \mathcal{O}\left(\log(n)^{2\lceil\eta\rceil} b_n^{2\eta} + \tfrac{1}{nb_n} + \log(n)^2 n^{-2(1\wedge\eta)}\right) \;\leq\; \mathcal{O}\left(\log(n)^{2\lceil\eta\rceil} n^{-\frac{1}{2}-r}\right).$$

Furthermore, for $t \leq 2L+1, \ldots, L + \lceil nb_n \rceil$, the local polynomial estimator effectively uses the smaller bandwidth $\frac{t-L}{n} \geq L$, hence

$$\|\hat{\psi}_{t,n} - \psi_{t-L,n}\|_{L_q}^2 = \mathcal{O}\left(\log(n)^{2\lceil\eta\rceil}(\tfrac{t-L_n}{n})^{2\eta} + \tfrac{1}{t-L_n} + \log(n)^2 n^{-2(1\wedge\eta)}\right).$$

Moreover, $\|\psi_{t-L,n} - \psi_{t,n}\|^2 = \mathcal{O}((L_n/n)^{2\eta})$, and hence $\|\max_{t=t_0,\ldots,n} |\hat{\psi}_{t,n} - \psi_{t,n}|\|_{L_q} = \mathcal{O}(n^{\frac{1}{q}-\epsilon}) + \mathcal{O}((L_n/n)^{\eta})$, for some $\epsilon > 0$, which tends to zero for q large enough. Thanks to this uniform convergence, we may restrict our attention to the event

$$A_n = \left\{\frac{1}{2}\psi_{t,n} \leq \hat{\psi}_{t,n} \leq 2\psi_{t,n}, \qquad \forall\, t = 1, \ldots, n\right\},$$

as $P(A_n) \to 1$ as $n \to \infty$. In this event, a Taylor expansion of m around $\hat{\psi}_{t,n}$ yields

$$\begin{aligned}
&\widehat{\mathcal{H}}(u) - \frac{1}{n}\sum_{t=1}^{\lfloor un \rfloor} m(\psi_{t,n}) \\
&= \frac{1}{n}\sum_{t=1}^{\lfloor un \rfloor}\left\{m(\hat{\psi}_{t,n}) + Dm(\hat{\psi}_{t,n}) \cdot (Z_{t,n} - \hat{\psi}_{t,n})\right\} - \frac{1}{n}\sum_{t=1}^{n} m(\psi_{t,n}) + \mathcal{O}_P\left(\tfrac{L_n}{n}\right) \\
&= \frac{1}{n}\sum_{t=1}^{\lfloor un \rfloor} Dm(\hat{\psi}_{t,n}) \cdot (Z_{t,n} - \psi_{t,n}) + \mathcal{O}_P\left(\frac{1}{n}\sum_{t=t_0}^{n} \|\hat{\psi}_{t,n} - \psi_{t,n}\|^2\right) + \mathcal{O}_P\left(\tfrac{L_n}{n}\right), \qquad (4)
\end{aligned}$$

using in (4) that the second derivative $D^2 m(\psi)$ is bounded in the specified neighborhood of $\psi_{t,n}$, and the special definition $\hat{\psi}_{t,n} = \psi_{t,n}$ for $t = 1, \ldots, t_0 - 1$. By Proposition 8 and our assumption on b_n and L_n, the latter term is of order $o_P(1/\sqrt{n})$. Moreover, the approximation (4) holds uniformly in $u \in [0, 1]$.

Next, note that the Hölder regularity of H_v with exponent η yields

$$\frac{1}{n}\sum_{t=1}^{\lfloor un \rfloor} m(\psi_{t,n}) = \frac{1}{n}\sum_{t=1}^{\lfloor un \rfloor} H_{t/n} = \int_0^u H_v \, dv + \mathcal{O}(n^{-\eta}),$$

uniformly in $u \in [0,1]$. Since $\eta > \frac{1}{2}$ by assumption, we have shown that

$$\sqrt{n}\left[\widehat{\mathcal{H}}(u) - \mathcal{H}(u)\right] = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor un \rfloor} Dm(\hat{\psi}_{t,n}) \cdot (Z_{t,n} - \psi_{t,n}) + o_P(1).$$

By Lemma 3, $|\mathbb{E}(Z_{t,n}) - \psi_{t,n}| = \mathcal{O}(\log(n) n^{-\eta}) = o(n^{-\frac{1}{2}})$, and we obtain

$$\sqrt{n}\left[\widehat{\mathcal{H}}(u) - \mathcal{H}(u)\right] = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor un \rfloor} Dm(\hat{\psi}_{t,n}) \cdot (Z_{t,n} - \mathbb{E}Z_{t,n}) + o_P(1). \tag{5}$$

Step (ii): We now apply the multiplier FCLT of Theorem 5 to the leading term in (5), with

$$\hat{g}_{t,n} = Dm(\hat{\psi}_{t,n}) \mathbb{1}_{A_n}, \qquad g_{t,n} = Dm(\psi_{t,n}), \qquad g_u = Dm(\psi_u), \qquad Y_{t,n} = Z_{t,n} - \mathbb{E}Z_{t,n}.$$

For this definition of $g_{t,n}$ and $\hat{g}_{t,n}$, (A.4) holds, Φ_n is bounded, and $\Psi_n = \mathcal{O}(n^{1-\eta})$. Moreover, Proposition 8 shows that

$$\begin{aligned}
\Lambda_n^2 &\le C \sum_{t=t_0}^{n} \left(\|\hat{\psi}_{t,n} - \psi_{t-L,n}\|_{L_2}^2 + \|\psi_{t-L,n} - \psi_{t,n}\|^2 \right) \\
&\le C \left(\log(n)^2 n^{\frac{1}{2}-r} + L_n^{2\eta} n^{1-2\eta} \right) + C \sum_{t=2L_n}^{L_n + \lceil nb_n \rceil} \left(\log(n)^2 (\tfrac{t-L_n}{n})^{2\eta} + \tfrac{1}{t-L_n} \right) \\
&\le C \left(\log(n)^2 n^{\frac{1}{2}-r} + L_n n^{1-2\eta} + \log(n)^2 \tfrac{(nb_n)^{2\eta+1}}{n^{2\eta}} + \log(nb_n) \right) \\
&= \mathcal{O}\left(\log(n)^2 n^{\frac{1}{2}-r} + L_n n^{1-2\eta} \right)
\end{aligned}$$

It remains to check the conditions (A.3), (A.1), (A.2) for $Y_{t,n}$. To write $Y_{t,n}$ in the form $G_{t,n}(\boldsymbol{\epsilon}_t)$ as in Appendix A, note that $Y_{t,n}$ is a functional of the driving Brownian motion B_s, see Definition (**??**). Since all Polish spaces are Borel-isomorphic, there exists a Borel-isomorphism $\varphi : (0,1) \to C[0,1]$, and iid random variables $\epsilon_t \sim U(0,1)$ such that $\varphi(\epsilon_t) = [\tilde{B}_u]_{u\in[0,1]} = \sqrt{n}[B_{\frac{u+t-1}{n}} - B_{\frac{t-1}{n}}]_{u\in[0,1]} = \varphi(\epsilon_t)$, for a standard Brownian motion $\tilde{B}$. With this notation, we may write

$$\begin{aligned}
\frac{n^{H_{i/n}}}{\sigma_{i/n}} \chi_{i,n} &= \frac{n^{H_{i/n}}}{\sigma_{i/n}} \int_{-\infty}^{i/n} g_{i,n}(s)\, dB_s = \widetilde{G}_{i,n}(\boldsymbol{\epsilon}_i), \\
g_{i,n}(s) &= \sigma_s \left[(\tfrac{i}{n} - s)_+^{H_s - \frac{1}{2}} - 2(\tfrac{i}{n} - \tfrac{1}{n} - s)_+^{H_s - \frac{1}{2}} + (\tfrac{i}{n} - \tfrac{2}{n} - s)_+^{H_s - \frac{1}{2}} \right], \\
\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_j) &= \frac{n^{H_{i/n}}}{\sigma_{i/n}} \int_{-\infty}^{i/n} g_{i,n}(s)\, dB_{s+\frac{j-i}{n}} \\
&= \frac{n^{H_{i/n}}}{\sigma_{i/n}} \sum_{k=0}^{\infty} \int_{(i-k-1)/n}^{(i-k)/n} g_{i,n}(s)\, dB_{s+\frac{j-i}{n}} \quad = \frac{n^{H_{i/n}}}{\sigma_{i/n}} \sum_{k=0}^{\infty} \int_{i-k-1}^{i-k} g_{i,n}(\tfrac{s}{n}) \tfrac{1}{\sqrt{n}} d\tilde{B}_{s+(j-i)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \int_{i-k-1}^{i-k} \tilde{g}_{i,n}(s)\, d\tilde{B}_{s+(j-i)} \\
&= \sum_{k=0}^{\infty} \zeta_{i,k,n}(\epsilon_{j-k}), \qquad \zeta_{i,k,n} : (0,1) \to \mathbb{R},\ \zeta_{i,k,n}(\epsilon_{j-k}) \in L_2(P), \\
\tilde{g}_{i,n}(s) &= \frac{\sigma_{\frac{s}{n}}}{\sigma_{\frac{i}{n}}} n^{H_{\frac{i}{n}} - H_{\frac{s}{n}}}\, \bar{g}(s-i, H_{\frac{s}{n}}), \\
\bar{g}_i(s,H) &= (-s)_+^{H-\frac{1}{2}} - 2(-1-s)_+^{H-\frac{1}{2}} + (-2-s)_+^{H-\frac{1}{2}}.
\end{aligned}
$$

In particular, the Bernoulli shift of the ϵ_t is equivalent to shifting the driving Brownian motion.

Ad (A.2): The $\zeta_{i,k,n}(\epsilon_j)$ are centered Gaussian random variables with variance

$$
\begin{aligned}
\|\zeta_{i,k,n}(\epsilon_j)\|_{L_2}^2 &= \int_{i-k-1}^{i-k} |\tilde{g}_{i,n}(s)|^2\, ds \\
&\leq C n^{2(H_{\frac{i}{n}} - H_{\frac{i-k}{n}}) + Cn^{-\eta}} (k+1)^{2H_{\frac{i-k}{n}} - 5 + Cn^{-\eta}} \\
&\leq C n^{2(H_{\frac{i}{n}} - H_{\frac{i-k}{n}})} (k+1)^{2H_{\frac{i-k}{n}} - 5 + \delta},
\end{aligned}
$$

for $\delta > 0$ small enough, and n large enough. If $H_{\frac{i}{n}} < H_{\frac{i-k}{n}}$, then $\|\zeta_{i,k,n}(\epsilon_j)\|_{L_2}^2 \leq C(k+1)^{5\overline{H}-5+\delta} \leq C(k+1)^{-3}$ for δ small enough, because $\overline{H} < 1$. If instead $H_{\frac{i}{n}} < H_{\frac{i-k}{n}}$, we further distinguish two cases to obtain

$$
\begin{aligned}
\|\zeta_{i,k,n}(\epsilon_j)\|_{L_2}^2 &\leq \begin{cases} C(k+1)^{\frac{2}{1-\delta}(H_{\frac{i}{n}} - H_{\frac{i-k}{n}})} (k+1)^{2H_{\frac{i-k}{n}} - 5 + \delta}, & k \geq n^{1-\delta}, \\ Cn^{Cn^{-\delta\eta}} (k+1)^{2H_{\frac{i-k}{n}} - 5 + \delta}, & k < n^{1-\delta}, \end{cases} \\
&\leq \begin{cases} C(k+1)^{\frac{2}{1-\delta} H_{\frac{i}{n}} + \frac{2\delta}{1-\delta} H_{\frac{i-k}{n}} - 5 + \delta}, & k \geq n^{1-\delta}, \\ C(k+1)^{2\overline{H} - 5 + \delta}, & k < n^{1-\delta}, \end{cases} \\
&\leq \begin{cases} C(k+1)^{2\frac{1+\delta}{1-\delta}\overline{H} - 5 + \delta}, & k \geq n^{1-\delta}, \\ C(k+1)^{2\overline{H} - 5 + \delta}, & k < n^{1-\delta}, \end{cases} \\
&\leq C(k+1)^{-3},
\end{aligned}
$$

for δ small enough. In all cases, we have shown that for all $q \geq 2$, and for some $C = C_q$,

$$
\begin{aligned}
\|\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_0) - \widetilde{G}_{i,n}(\tilde{\boldsymbol{\epsilon}}_{0,h})\|_{L_q} \leq C \|\zeta_{i,h,n}(\epsilon_j)\|_{L_2} &\leq C(h+1)^{-\frac{3}{2}}, \\
\|\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_0)\|_{L_q} &\leq C.
\end{aligned}
$$

Now note that $Y_{t,n}$ is a function of the $\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_i)$, in particular

$$
Y_{i,n} = G_{t,n}(\boldsymbol{\epsilon}_j) = \begin{pmatrix} \widetilde{G}_{t,n}(\boldsymbol{\epsilon}_j)^2 \\ (\widetilde{G}_{t,n}(\boldsymbol{\epsilon}_j) + 2\widetilde{G}_{t-1,n}(\boldsymbol{\epsilon}_{j-1}) + \widetilde{G}_{t-2,n}(\boldsymbol{\epsilon}_{j-2}))^2 \end{pmatrix}, \tag{6}
$$

and it is straightforward to derive

$$\|G_{t,n}(\boldsymbol{\epsilon}_0) - G_{t,n}(\tilde{\boldsymbol{\epsilon}}_{0,h})\|_{L_q} \leq C(h+1)^{-\frac{3}{2}}.$$

This establishes (A.2) with $\Theta_n = \mathcal{O}(1)$ and exponent $\beta = \frac{3}{2}$.
Ad (A.1): Observe that

$$\begin{aligned}
&\left\|\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_0) - \widetilde{G}_{i-1,n}(\boldsymbol{\epsilon}_0)\right\|_{L_2}^2 \\
&= \sum_{k=0}^{\infty} \int_{i-k-1}^{i-k} |\tilde{g}_{i,n}(s) - \tilde{g}_{i-1,n}(s-1)|^2 \, ds \\
&\leq Cn^{-2\eta} n^{Cn^{-\eta}} \sum_{k=0}^{\infty} (k+1)^{-3} \qquad (7) \\
&\quad + Cn^{Cn^{-\eta}} \sum_{k=0}^{\infty} \int_{i-k-1}^{i-k} |\bar{g}(s-i, H_{\frac{s}{n}}) - \bar{g}(s-i, H_{\frac{s-1}{n}})|^2 \, ds \\
&\leq Cn^{-2\eta} + \sum_{k=0}^{\infty} \int_{-k-1}^{-k} |\bar{g}(s, H_{\frac{s}{n}}) - \bar{g}(s, H_{\frac{s-1}{n}})|^2 \, ds, . \qquad (8)
\end{aligned}$$

To bound the latter integral, we exploit the fact that

$$\begin{aligned}
\frac{d}{dH}\bar{g}(s,H) &= \mathbb{1}_{s<0} \log(-s)(-s)^{H-\frac{1}{2}} \\
&\quad - 2\mathbb{1}_{s<-1} \log(-s-1)(-s-1)^{H-\frac{1}{2}} \\
&\quad + \mathbb{1}_{s<2} \log(-s-2)(-s-2)^{H-\frac{1}{2}}, \\
\rightsquigarrow \quad \left|\frac{d}{dH}\bar{g}(s,H)\right| &\leq C(1+|\log s|) \min\left(|s|^{H-\frac{1}{2}}, |s|^{H-\frac{5}{2}}\right).
\end{aligned}$$

Hence, the mean value theorem yields some $\tilde{H}_s$ between $H_{\frac{s}{n}}$ and $H_{\frac{s-1}{n}}$ such that

$$\begin{aligned}
\int_{-k-1}^{-k} |\bar{g}(s, H_{\frac{s}{n}}) - \bar{g}(s, H_{\frac{s-1}{n}})|^2 \, ds &\leq \int_{-k-1}^{-k} |H_{\frac{s}{n}} - H_{\frac{s-1}{n}}|^2 \left|\frac{d}{dH}\bar{g}_{0,n}(s,H)|_{H=\tilde{H}_s}\right|^2 \, ds \\
&\leq Cn^{-2\eta} \int_{-k-1}^{-k} [1+|\log s|] \left[\mathbb{1}_{|s|<1}|s|^{2\underline{H}-1} + \mathbb{1}_{|s|\geq 1}|s|^{2\overline{H}-5}\right], ds \\
&\leq Cn^{-2\eta}(k+1)^{-3}.
\end{aligned}$$

Using this bound in (8), and exploiting the Gaussianity, we obtain for all $q \geq 2$

$$\left\|\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_0) - \widetilde{G}_{i-1,n}(\boldsymbol{\epsilon}_0)\right\|_{L_q}^2 \leq Cn^{-2\eta}.$$

Since $\|\tilde{G}_{i,n}\|_{L_q}$ is bounded, it is straightforward to conclude from this and (6) that

$$\|G_{i,n}(\boldsymbol{\epsilon}_0) - G_{i-1,n}(\boldsymbol{\epsilon}_0)\|_{L_2}^2 \leq Cn^{-2\eta},$$

and hence (A.1) holds with $\Theta_n = \mathcal{O}(1)$ and $\Gamma_n = n^{1-\eta}$.
Ad (A.3): For $u \in [0,1]$, define the limiting kernels

$$\widetilde{G}_u(\boldsymbol{\epsilon}_j) = \int_{-\infty}^{0} \bar{g}(s, H_u)\, d\tilde{B}_{s+j},$$

and, in view of (6),

$$G_u(\boldsymbol{\epsilon}_j) = \left(\frac{\widetilde{G}_u(\boldsymbol{\epsilon}_j)^2}{\left(\widetilde{G}_u(\boldsymbol{\epsilon}_j) + 2\widetilde{G}_u(\boldsymbol{\epsilon}_{j-1}) + \widetilde{G}_u(\boldsymbol{\epsilon}_{j-2})\right)^2}. \right)$$

Then

$$\begin{aligned}
\|\widetilde{G}_{\lfloor un\rfloor,n}(\boldsymbol{\epsilon}_0) - \widetilde{G}_u(\boldsymbol{\epsilon}_0)\|_{L_q} &\le C\|\widetilde{G}_{\lfloor un\rfloor,n}(\boldsymbol{\epsilon}_0) - \widetilde{G}_u(\boldsymbol{\epsilon}_0)\|_{L_2} \\
&\le \int_{-\infty}^{\lfloor un\rfloor} |\bar{g}(s - \lfloor un\rfloor, H_u) - \tilde{g}_{\lfloor un\rfloor,n}(s)|^2\, ds \\
&= \int_{-\infty}^{0} \left| \bar{g}(s, H_u) - \frac{\sigma_{\frac{\lfloor un\rfloor+s}{n}}}{\sigma_{\frac{\lfloor un\rfloor}{n}}} n^{H_{\frac{\lfloor un\rfloor}{n}} - H_{\frac{\lfloor un\rfloor+s}{n}}} \bar{g}(s, H_{\frac{\lfloor un\rfloor+s}{n}}) \right|^2 ds \\
&\le 2\int_{-\infty}^{0} \left| \bar{g}(s, H_u) - \bar{g}(s, H_{\frac{\lfloor un\rfloor+s}{n}}) \right|^2 ds \\
&\quad + 2\int_{-\infty}^{0} \left| 1 - \frac{\sigma_{\frac{\lfloor un\rfloor+s}{n}}}{\sigma_{\frac{\lfloor un\rfloor}{n}}} n^{H_{\frac{\lfloor un\rfloor}{n}} - H_{\frac{\lfloor un\rfloor+s}{n}}} \right|^2 \left| \bar{g}(s, H_{\frac{\lfloor un\rfloor+s}{n}}) \right|^2 ds \\
&\le Cn^{-2\eta} + C\int_{-\infty}^{-1} \left[(\tfrac{|s|}{n})^\eta \wedge 1 \right]^2 \sup_{H\in[\underline{H},\overline{H}]} \left| \tfrac{d}{dH}\bar{g}(s,H) \right|^2 ds \\
&\quad + C\log(n) \int_{-\infty}^{-1} \left[(\tfrac{|s|}{n})^\eta \wedge 1 \right]^2 \sup_{H\in[\underline{H},\overline{H}]} |\bar{g}(s,H)|^2\, ds \\
&\le Cn^{-2\eta} + C\log(n) n^{-2\eta} \int_1^n |s|^{2\overline{H}+2\eta-5}\, ds + C\log(n)\int_n^\infty |s|^{2\overline{H}-5}\, ds \\
&\le C\log(n) n^{-2\eta}
\end{aligned}$$

because $|\bar{g}(s,H)| \le C|s|^{H-\frac{5}{2}}$ and $\frac{d}{dH}|\bar{g}(s,H)| \le C(1+\log|s|)|s|^{H-\frac{5}{2}}$ for $|s|\ge 1$. That is, upon choosing $\delta > 0$ small enough, we find that for any $q \ge 2$

$$\|\widetilde{G}_{\lfloor un\rfloor,n}(\boldsymbol{\epsilon}_0) - \widetilde{G}_u(\boldsymbol{\epsilon}_0)\|_{L_q} \le Cn^{-\delta}.$$

Again, this directly yields that

$$\|G_{\lfloor un\rfloor,n}(\boldsymbol{\epsilon}_0) - G_u(\boldsymbol{\epsilon}_0)\|_{L_q} \le Cn^{-\delta},$$

and hence $\int_0^1 \|G_{\lfloor un\rfloor,n}(\boldsymbol{\epsilon}_0) - G_u(\boldsymbol{\epsilon}_0)\|_{L_2}\, du \to 0$ as $n \to \infty$. This establishes (A.3).

Rate constraints: In the previous steps, we have verified the conditions of Theorem 5 for any $q \geq 2$, for $\beta = \frac{3}{2}$, and with $\Gamma_n \asymp n^{1-\eta}$, $\Lambda_n^2 \asymp \log(n)^2 n^{\frac{1}{2}-r} + L_n^{2\eta} n^{1-2\eta}$, $\Psi_n \asymp n^{1-\eta}$, $\Phi_n = \mathcal{O}(1)$, $\Theta_n = \mathcal{O}(1)$. Upon choosing q large enough, we have $\xi(q,\beta) = \frac{\beta-1}{4\beta-2} = \frac{1}{8}$, hence

$$(\Gamma_n + \Psi_n)^{\frac{\beta-1}{2\beta}} \sqrt{\log(n)} n^{-\xi(q,\beta)} = n^{\frac{1-\eta}{6}} \sqrt{\log(n)} n^{-\frac{1}{8}},$$

which tends to zero if $\eta > \frac{1}{4}$. Moreover, upon choosing q large enough, we have

$$\begin{aligned} & n^{-\frac{1}{2}} \Lambda_n^2 + \Lambda_n L_n^{-\beta} + n^{\frac{1}{q}-\frac{1}{2}} L_n \to 0 \\ \Longleftarrow n^{\frac{1}{6}-\frac{r}{3}} \ll L_n \ll n^{\frac{1}{2}-\delta} \quad & \text{for some } \delta > 0, \text{ and } L_n \ll n^{1-\frac{1}{4\eta}}. \end{aligned}$$

Thus, we have shown that Theorem 5 is applicable under the conditions formulated in Theorem 3.

Determining the asymptotic variance: The asymptotic local variance is given by

$$\begin{aligned} \Sigma_u &= \sum_{h=-\infty}^{\infty} g_u \operatorname{Cov}(G_u(\boldsymbol{\epsilon}_0), G_u(\boldsymbol{\epsilon}_h)) g_u^T, \\ g_u &= \frac{1}{2\,\Gamma_{H_u}(0)} \begin{pmatrix} -1 \\ 2^{-2H_u} \end{pmatrix}. \end{aligned}$$

The autocovariances may be computed as in Lemma 4, hence $\operatorname{Cov}(G_u(\boldsymbol{\epsilon}_0), G_u(\boldsymbol{\epsilon}_h)) = \Sigma_{H_u}(h)$ as therein. Thus,

$$\begin{aligned} \Sigma_u &= \tau^2(H_u), \\ \tau^2(H) &= \frac{1}{4\Gamma_H(0)^2} \sum_{h=-\infty}^{\infty} (-1, 2^{-2H}) \Sigma_H(h) \begin{pmatrix} -1 \\ 2^{-2H} \end{pmatrix} \\ &= \frac{1}{2\Gamma_H(0)^2} \sum_{h=-\infty}^{\infty} \Big\{ \Gamma_H(h)^2 + 2^{-4H}(2\Gamma_H(h) + \Gamma_H(h-1) + \Gamma_H(h+1))^2 \\ &\qquad - 2^{-2H} \left[(\Gamma_H(h) + \Gamma_H(h-1))^2 + (\Gamma_H(h) + \Gamma_H(h-1))^2 \right] \Big\} \\ &= \frac{1}{2\Gamma_H(0)^2} \sum_{h=-\infty}^{\infty} \Big\{ \Gamma_H(h)^2 + 2^{-4H}(2\Gamma_H(h) + \Gamma_H(h-1) + \Gamma_H(h+1))^2 \\ &\qquad - 2^{-2H+1} (\Gamma_H(h) + \Gamma_H(h-1))^2 \Big\}. \end{aligned} \tag{9}$$

$\square$

Proof of Theorem 4. By Theorem 1, and as established in the proof of Theorem 3, we have for any $q \geq 2$

$$\max_{t=2L_n,\dots,n} |\widehat{H}_n(\tfrac{t}{n}) - H_{\frac{t}{n}}| \quad \xrightarrow{\mathbb{P}} \quad 0.$$

Hence,

$$\sup_{u\in[0,1]}\left|\frac{1}{n}\sum_{t=2L}^{\lfloor un\rfloor}\tau^2(H(\tfrac{t}{n}))-\frac{1}{n}\sum_{t=2L}^{\lfloor un\rfloor}\tau^2(\widehat{H}_n(\tfrac{t}{n}))\right|\xrightarrow{\mathbb{P}}0.$$

Moreover, continuity of $v\mapsto H_v$, and $L_n/n\to 0$, yields

$$\sup_{u\in[0,1]}\left|\int_0^u\tau^2(H_v)\,dv-\frac{1}{n}\sum_{t=2L_n}^{\lfloor un\rfloor}\tau^2(H(\tfrac{t}{n}))\right|\to 0.$$

□

Proof of Proposition 1. Convergence under the null is a direct consequence of Theorems 3 and 4. Under the alternative, the function $u\mapsto\mathcal{H}(u)$ is not linear. This holds because $v\mapsto H_v$ is continuous, and hence H_v can not deviate at a single point v. Thus, $\sup_{u\in[0,1]}|\mathcal{H}(u)-u\mathcal{H}(1)|>0$. Moreover, Theorem 3 yields $\|\widehat{\mathcal{H}}-\mathcal{H}\|_\infty=\mathcal{O}(1/\sqrt{n})$ and thus $\sqrt{n}T_{\mathrm{CUSUM}}(\widehat{\mathcal{H}})\to\infty$ in probability, at rate $\sqrt{n}$. On the other hand, $q_n(\alpha)\xrightarrow{\mathbb{P}}q_(\alpha)$ as a consequence of Theorem 4. This establishes consistency of the test. □

Proof of Proposition 2. The first claim holds because $q_n(\alpha)\to q(\alpha)$ by Theorem 4, where $q(\alpha)$ is the $1-\alpha$ quantile of $Z=\sup_u|W(\int_0^u\tau^2(H_v)\,dv)|$, and $\sqrt{n}\widehat{T}(\mathcal{G}_0)\le\sqrt{n}\widehat{T}(H)\Rightarrow Z$.

We proceed to prove the second claim. If $\mathcal{G}_0$ is closed, then $\widetilde{\mathcal{G}}_0=\{u\mapsto\int_0^u\tilde{H}(v)\,dv\mid\tilde{H}\in\mathcal{G}_0\}$ is also closed, and $\int H(v)\,dv\notin\widetilde{\mathcal{G}}_0$. Hence, $\sup_{u\in[0,1]}|\int_0^u\tilde{H}(v)\,dv-\int_0^u\tilde{H}(v)\,dv|>0$ for all $\tilde{H}\in\mathcal{G}_0$. As $\sqrt{n}(\widehat{\mathcal{H}}-\mathcal{H})=o_P(1)$, this implies that $\sqrt{n}\widehat{T}(\mathcal{G}_0)\xrightarrow{\mathbb{P}}\infty$. On the other hand, Theorem 4 yields convergence of $q_n(\alpha)$ to a finite number, establishing consistency of the test. □

References

Ayache, A., Bertrand, P., and Véhel, J. L. (2007). A Central Limit Theorem for the Generalized Quadratic Variation of the Step Fractional Brownian Motion. *Statistical Inference for Stochastic Processes*, 10(1):1–27.

Ayache, A. and Bouly, F. (2021). Moving average Multifractional Processes with Random Exponent: Lower bounds for local oscillations. *Stochastic Processes and their Applications*, 146:143–163.

Ayache, A. and Bouly, F. (2023). Uniformly and strongly consistent estimation for the random Hurst function of a multifractional process. *ALEA*, 20:1587–1614. Publisher: Unpublished.

Ayache, A., Esser, C., and Hamonier, J. (2018). A new multifractional process with random exponent. *Risk and Decision Analysis*, 7(1-2):5–29.

Ayache, A. and Lévy Véhel, J. (2004). On the identification of the pointwise Hölder exponent of the generalized multifractional Brownian motion. *Stochastic Processes and their Applications*, 111(1):119–156.

Bardet, J.-M. (2018). Theoretical and Numerical Comparisons of the Parameter Estimator of the Fractional Brownian Motion.

Bardet, J.-M. and Surgailis, D. (2013). Nonparametric estimation of the local Hurst function of multifractional Gaussian processes. *Stochastic Processes and their Applications*, 123(3):1004–1045.

Benassi, A., Bertrand, P., Cohen, S., and Istas, J. (2000). Identification of the Hurst Index of a Step Fractional Brownian Motion. *Statistical Inference for Stochastic Processes*, 3(1/2):101–111.

Bertrand, P. R., Combes, J.-L., Dury, M.-E., and Hadouni, D. (2018). Overfitting of Hurst estimators for multifractional Brownian motion: A fitting test advocating simple models. *Risk and Decision Analysis*, 7(1-2):31–49.

Bertrand, P. R., Fhima, M., and Guillin, A. (2013). Local estimation of the Hurst index of multifractional Brownian motion by increment ratio statistic method. *ESAIM: Probability and Statistics*, 17:307–327.

Bianchi, S., Pantanella, A., and Pianese, A. (2013). Modeling stock prices by multifractional Brownian motion: an improved estimation of the pointwise regularity. *Quantitative Finance*, 13(8):1317–1330.

Bibinger, M. (2020). Cusum tests for changes in the Hurst exponent and volatility of fractional Brownian motion. *Statistics & Probability Letters*, 161:108725.

Brouste, A. and Fukasawa, M. (2018). Local asymptotic normality property for fractional Gaussian noise under high-frequency observations. *The Annals of Statistics*, 46(5):2045–2061.

Chong, C. H. and Todorov, V. (2025). A Nonparametric Test for Rough Volatility. *Journal of the American Statistical Association*, pages 1–23.

Coeurjolly, J.-F. (2001). Estimating the Parameters of a Fractional Brownian Motion by Discrete Variations of its Sample Paths. *Statistical Inference for Stochastic Processes*, 4(2):199–227.

Coeurjolly, J.-F. (2005). Identification of multifractional Brownian motion. *Bernoulli*, 11(6):987–1008.

Cohen, S., Gamboa, F., Lacaux, C., and Loubes, J.-M. (2013). LAN property for some fractional type Brownian motion. Technical Report 1.

Cui, Y., Levine, M., and Zhou, Z. (2021). Estimation and inference of time-varying autocovariance under complex trend: A difference-based approach. *Electronic Journal of Statistics*, 15(2).

Demetrescu, M. and Wied, D. (2018). Testing for constant correlation of filtered series under structural change. *Econometrics Journal*, 22:10–33.

Dette, H., Wu, W., and Zhou, Z. (2019). Change Point Analysis of Correlation in Non-stationary Time Series. *Statistica Sinica*, 29:611–643. arXiv: 1801.10478.

Fan, J. and Gijbels, I. (2018). *Local Polynomial Modelling and Its Applications*. Routledge, 1 edition.

Garcin, M. (2017). Estimation of time-dependent Hurst exponents with variational smoothing and application to forecasting foreign exchange rates. *Physica A: Statistical Mechanics and its Applications*, 483:462–479.

Gatheral, J., Jaisson, T., and Rosenbaum, M. (2018). Volatility is rough. *Quantitative finance*, 18(6):933–949.

Górecki, T., Horváth, L., and Kokoszka, P. (2018). Change point detection in heteroscedastic time series. *Econometrics and Statistics*, 7:63–88. arXiv: 1612.02794v1.

Herbin, E. (2006). From N Parameter Fractional Brownian Motions to N Parameter Multifractional Brownian Motions. *Rocky Mountain Journal of Mathematics*, 36(4):1249–1284. arXiv: math/0503182v1.

Istas, J. and Lang, G. (1997). Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, 33(4):407–436.

Kent, J. T. and Wood, A. T. (1997). Estimating the fractal dimension of a locally self-similar gaussian process by using increments. *Journal of the Royal Statistical Society. Series B: Statistical Methodology*, 59(3):679–699.

Lebovits, J. and Podolskij, M. (2017). Estimation of the global regularity of a multifractional Brownian motion. *Electronic Journal of Statistics*, 11(1):78–98.

Loboda, D., Mies, F., and Steland, A. (2021). Regularity of multifractional moving average processes with random Hurst exponent. *Stochastic Processes and their Applications*, 140:21–48.

Mies, F. (2021). Functional estimation and change detection for nonstationary time series. *Journal of the American Statistical Association*, to appear.

Mies, F. (2024). Strong Gaussian approximations with random multipliers. arXiv:2412.14346 [math].

Nourdin, I. and Peccati, G. (2012). *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality*. Number 192 in Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge [England] ; New York.

Peltier, R.-F. and Lévy Véhel, J. (1995). Multifractional Brownian Motion: Definition and Preliminary Results. Technical report, INRIA.

Pesta, M. and Wendler, M. (2020). Nuisance-parameter-free changepoint detection in non-stationary series. *TEST*, 29:379–408. arXiv: 1808.01905.

Pianese, A., Bianchi, S., and Palazzo, A. M. (2018). Fast and unbiased estimator of the time-dependent Hurst exponent. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 28(3):031102.

Schmidt, S. K., Wornowizki, M., Fried, R., and Dehling, H. (2021). An asymptotic test for constancy of the variance under short-range dependence. *The Annals of Statistics*, 49(6).

Shen, J. and Hsing, T. (2020). Hurst function estimation. *The Annals of Statistics*, 48(2):838–862.

Stoev, S. A. and Taqqu, M. S. (2006). How rich is the class of multifractional Brownian motions? *Stochastic Processes and their Applications*, 116(2):200–221.

Tsybakov, A. B. (2008). *Introduction to Nonparametric Estimation*, volume 53. Springer.

Zhou, Z. (2013). Heteroscedasticity and autocorrelation robust structural change detection. *Journal of the American Statistical Association*, 108(502):726–740.