Sharp multiscale change detection for time series

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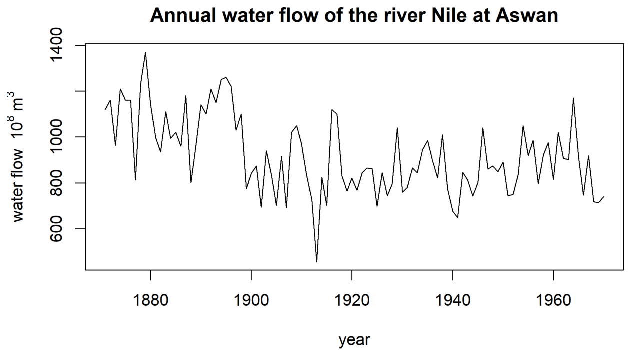
11th Bernoulli-IMS World Congress
2024-08-13, Bochum

Changepoint example



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H_1 : the process is not stationary

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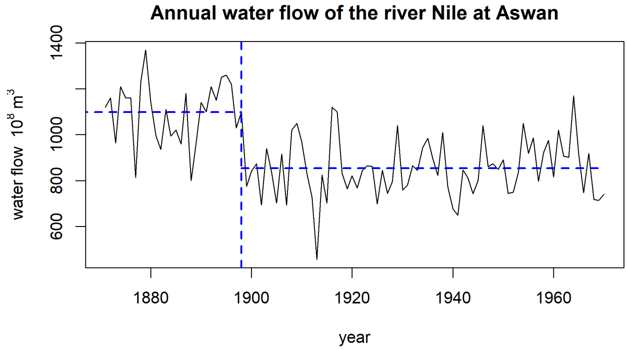

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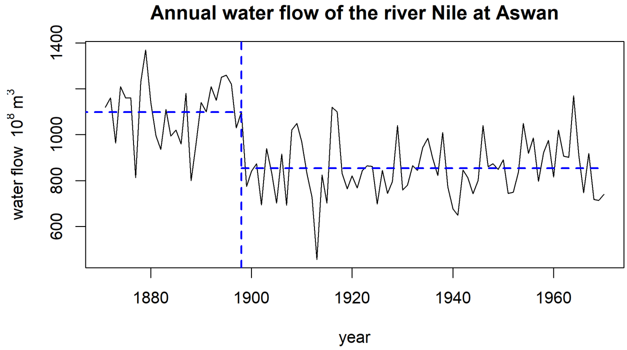

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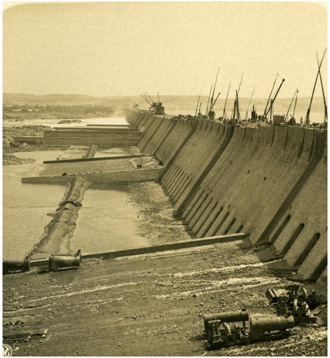

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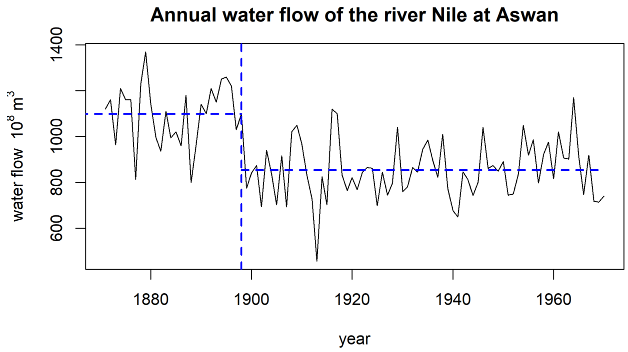

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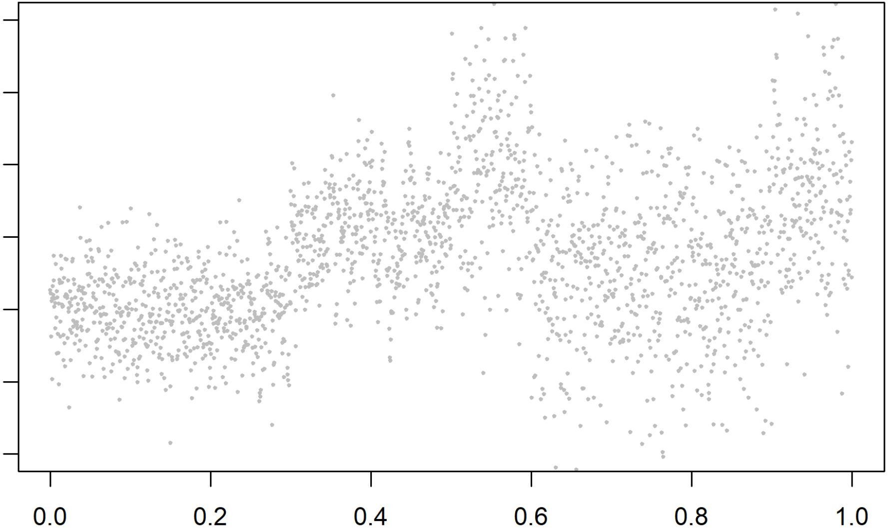
Statistically similar applications:

- Trend changes in house prices
- Volatility change in a portfolio
- Faults in a wind turbine
- Quality degradation in a manufacturing system

How many structural breaks?



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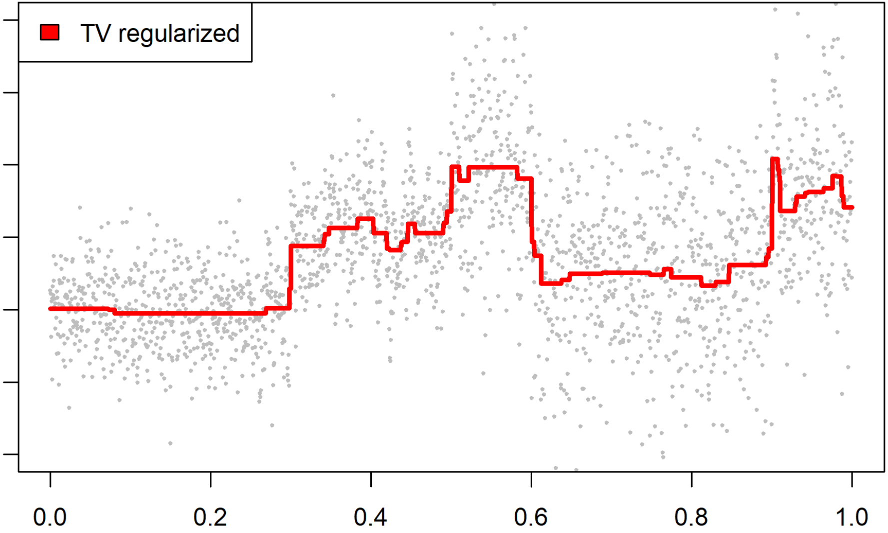

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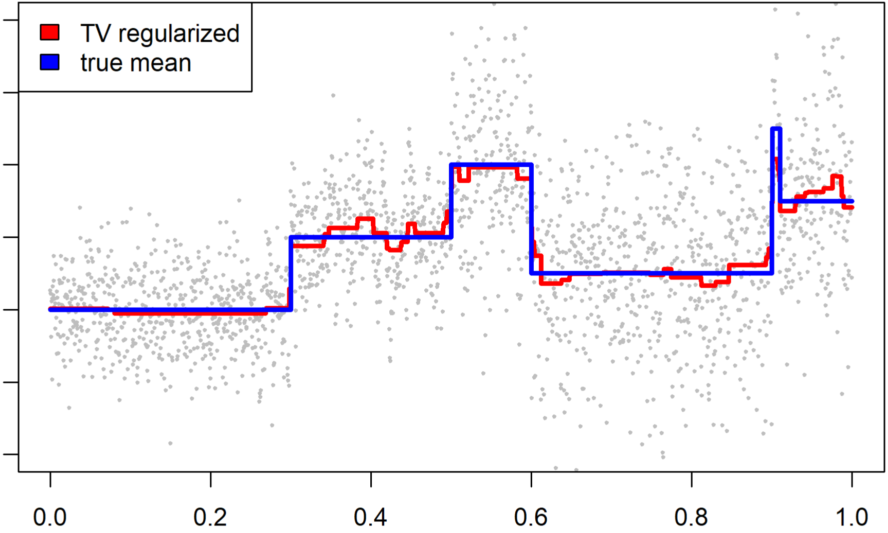

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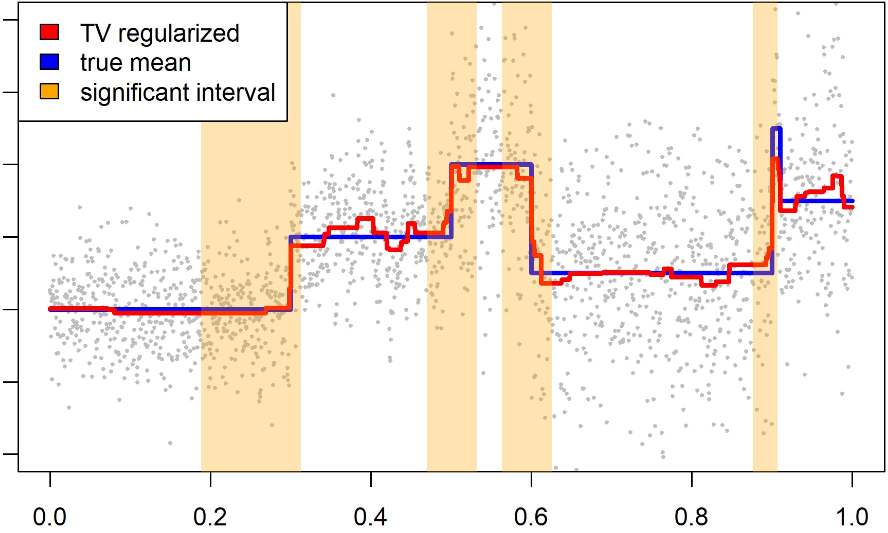


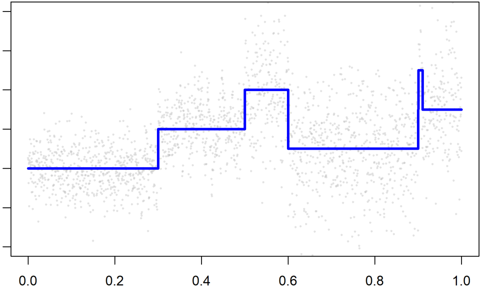
Multiple changes in mean

Multiple change point problem

We observe $X_1, \ldots, X_n$ where

$$X_t = \epsilon_t + \mu_t,$$
$$\mu_t = \sum_{k=1}^{m} \mu^{(k)} \mathbb{1}(t \in (\tau_{k-1}, \tau_k]).$$

for centered error terms ϵ_t (typically iid).



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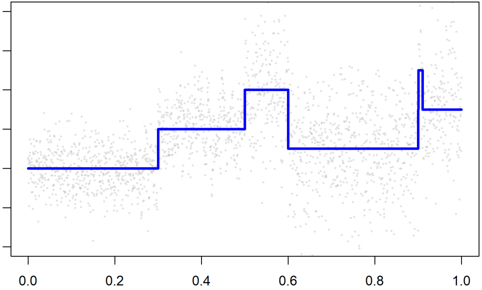

- $\Delta_k = \mu_k - \mu_{k-1}$ (size of change)
- $L_k = |\tau_k - \tau_{k-1}| \wedge |\tau_{k+1} - \tau_k|$ (length of change)

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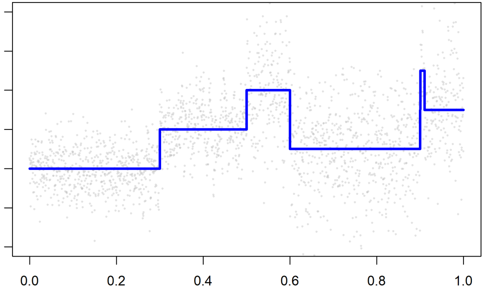

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Detectability (Verzelen et al., 2023)

Changepoint τ_k is detectable if and only if

$$E_k = \Delta_k \sqrt{L_k} \gg \sqrt{\log \frac{n}{L_k}}$$

Multiscale scan statistic for changepoints

$$T_n = \sup_{I \subset [1,n]} \inf_{\mu} \left[\overbrace{\sup_{(u,v] \subset I} \left| \frac{1}{\sqrt{|v-u|}} \sum_{t=u+1}^{v} (X_t - \mu) \right|}^{\text{scan statistic for interval } (u,v]} - \overbrace{c_\alpha \sqrt{\log \frac{e\,n}{|v-u|}}}^{\text{multiple testing correction}} \right]$$

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- $P(T_n > 0) \leq \alpha$ under H_0,
- $P(T_n > 0) \to 1$ if there is at least one detectable change

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Our contribution

Derive a procedure which is

- feasible
- optimal
- for dependent, nonstationary errors

via novel asymptotic theory.

Rewrite the statistic as a Hölder norm

$$T_n = \sup_{I\subset[1,n]} \inf_{\mu} \left[\sup_{(u,v]\subset I} \left| \frac{1}{\sqrt{|v-u|}} \sum_{t=u+1}^{v} (X_t - \mu) \right| - c_\alpha \sqrt{\log \frac{e\,n}{|v-u|}} \right]$$

Step (i): bound the test statistic under H_0 as

$$T_n \leq T_n^* = \sup_{0\leq u<v\leq 1} \frac{|S_n(v) - S_n(u)|}{\sqrt{|u-v|}} - c_\alpha \sqrt{\log \frac{e}{|u-v|}}, \qquad \text{where } S_n(u) = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor un \rfloor} \epsilon_t + \frac{u - \frac{\lfloor un \rfloor}{n}}{\sqrt{n}} \epsilon_{\lceil un \rceil}.$$

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Step (ii): Reformulate the decision rule as

$$\overline{T}_n := \sup_{0 \leq u < v \leq 1} \frac{|S_n(v) - S_n(u)|}{\sqrt{|u-v| \log(e/|u-v|)}} > c_\alpha$$

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Step (iii): Rewrite as Hölder norm with modulus $\rho_0(h) = \sqrt{h(1 + \log h)}$

$$\overline{T}_n := \|S_n\|_{\rho_0} \xrightarrow{d} \|\sigma B\|_{\rho_0} \quad ?$$

Asymptotic critical values

Theorem

- Let $S_n \xrightarrow{d} W$ in $C_0[0,1]$, for a Gaussian process W, and (Donsker's theorem)
- the increments of S_n are sub-Gaussian, $\|S_n(u) - S_n(v)\|_{\psi_2} \leq C\sqrt{|u-v|}$. (tightness)

Then there exists a $t_0 > 0$ such that

$$P\left(\|S_n\|_{\rho_0} > t\right) \to P\left(\|W\|_{\rho_0} > t\right) \quad \forall t \geq t_0.$$

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Question 1: Is this just weak convergence?

- $\rho_0(h) = \sqrt{h \log(e/h)}$ is exactly the modulus of continuity of a Brownian motion
- Brownian motion belongs to C^{ρ_0}, but its probability measure is **not tight**.
- However, we do obtain weak convergence in the Hölder space C^{ρ}, for all $\rho(h) \gg \rho_0(h)$.

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- Let $X_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$,
- and $X_t' = \sqrt{2}N_t \cdot X_t$, for $N_t \overset{\text{iid}}{\sim} \text{Ber}(0.5)$
- We have $S_n \xrightarrow[n\to\infty]{d} W$ and $S_n' \xrightarrow[n\to\infty]{d} W$, but X_t' has a larger sub-Gaussian norm.

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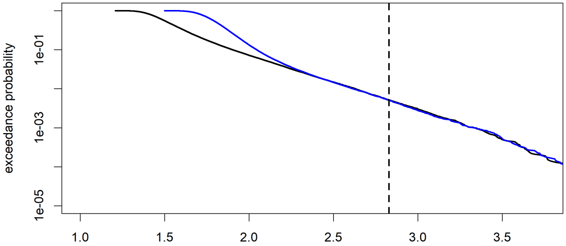

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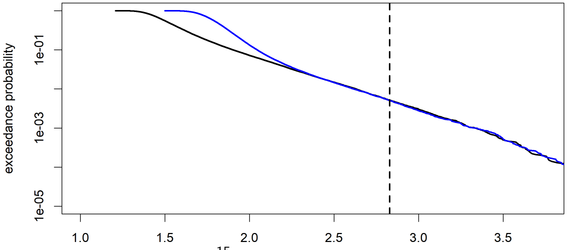

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Statistical implication:

- Use multiscale statistic with critical value c_α such that $P(\|W\|_{\rho_0} > c_\alpha) \leq \alpha$
- If α is small enough, the test will be asymptotically valid.

Bootstrapping critical values

What is the limit W?

If the errors ϵ_t are suitably ergodic and locally-stationary, then

$$S_n(u) \xrightarrow{d} W(u) = \int_0^u \sigma_s \, dB_s$$

and σ_s^2 is the local long-run-variance.

Bootstrapping critical values

Approximate W based on $\eta_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$ as

$$W_n^*(u) = \frac{1}{\sqrt{n}} \sum_{t=b}^{\lfloor nu \rfloor} \left(\frac{1}{\sqrt{b}} \sum_{s=t-b+1}^{t} X_t - \widehat{\mu}_t \right) \cdot \eta_t,$$

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Bootstrap consistency

- If $b = b_n$ and c_n are chosen suitably, and
- $\frac{1}{n}\sum_{t=1}^n |\hat{\mu}_t - \mu_t|^2 = \mathcal{O}(n^{-\eta})$ for some $\eta > 0$,

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(i) Compute the multiscale test statistic T_n
(ii) Estimate μ_t nonparametrically
(iii) Sample T_n^* repeatedly as above, to obtain an approximate quantile $\hat{c}_\alpha$
(iv) Detect a change if $T_n > \hat{c}_\alpha$.

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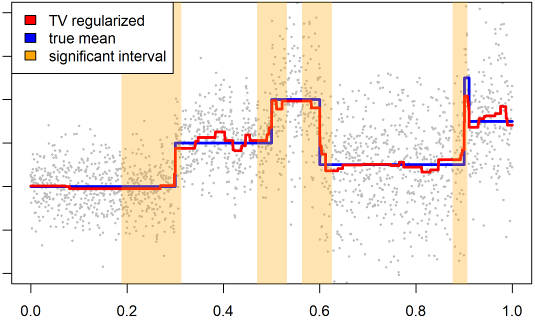
Intervals of significance



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- As in Fryzlewicz (2023), we find that with probability at least $1 - \alpha$, every $\widehat{I}_k$ contains at least one changepoint:

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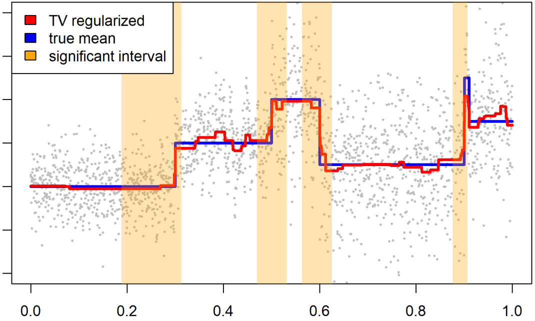

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Proposition (Power)

If the k-th changepoint τ_k satisfies $\Delta_k \sqrt{L_k} \gg \sqrt{\log(n/L_k)}$, then

$$P(T_n(I) > \hat{c}_\alpha) \to 1.$$

Technicalities: Increments of the partial sum process

Recall: We require the interpolated partial sum process $S_n(u)$ to satisfy

(i) $S_n \Rightarrow W$ in $C_0(0,1)$

(ii) $\|S_n(u) - S_n(v)\|_{\psi_2} \leq C\sqrt{|u-v|}$

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Assumption (Local stationarity)

The noise process is given by

$$\epsilon_t = \epsilon_{t,n} = G_{t/n}(\eta_t, \eta_{t-1}, \ldots)$$

for $\eta_i \overset{\text{iid}}{\sim} U(0,1)$, and the mapping $u \mapsto G_u(\eta)$ has bounded variation, measured in $\|\cdot\|_{\psi_2}$.

Definition

For any $h \in \mathbb{N}$, define the sub-Gaussian physical dependence measure of $(\epsilon_t)_{t\in\mathbb{Z}}$ as

$$\delta_{\psi_2}(h) := \sup_u \left\| G_u(\eta_t) - G_u(\eta_{t,h}) \right\|_{\psi_2}.$$

- $\delta_{\psi_2}(h) = \mathcal{O}(h^{-2-\eta})$ is sufficient for $S_n \Rightarrow W$.

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Theorem

$$\|S_n(u) - S_n(v)\|_{\psi_2} \leq C\sqrt{|u-v|}\sum_{j=1}^{\infty}\sqrt{j}\delta_{\psi_2}(j)$$

Technicalities: Bootstrap consistency

Theorem

Suppose that

- the noise ϵ_t is locally stationary, and its physical dependence measure decays as $\delta_{\psi_2}(j) = \mathcal{O}(j^{-3})$, and
- the mean estimator satisfies $\frac{1}{n}\sum_{t=1}^{n} |\hat{\mu}_t - \mu_t|^2 = \mathcal{O}_{L_p}(n^{-\eta})$.

then the 2-Wasserstein distance between T_n and the (X-conditional) distribution of T_n^* is bounded as

$$d_{W_2}(T_n, T_n^*)^2 = \frac{\log(n)}{\rho_0(c_n)}\, \mathcal{O}_P\left(\left(\frac{b}{n}\right)^{\frac{1}{4}} + \frac{1}{\sqrt{b}} + n^{-\eta/4}\right).$$

Summary

Methodological findings:

(i) Asymptotic threshold c_α for multiscale statistic only depends on second moments

(ii) Sub-Gaussian variance determines for which significance levels α the asymptotic threshold is applicable.

(iii) Consider smallest scales of the data, but neglect them when bootstrapping.

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Methodological findings:

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Thank you for your attention!

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